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OPTIMAL PRICE DYNAMICS AND SPECULATION WITH A STORABLE GOOD

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This paper analyzes as a dynamic game the optimal price and storage strategies of, respectively: (a) the seller of a storable good, who must keep pace with inflation but incurs a cost to changing his price; (b) his customers, who speculate on the timing of price adjustments to buy and store just before. A unique (Markov) perfect equilibrium is shown to exist, and is fully characterized. It generally involves a phase of mixed strategies, during which the seller tries to deter speculation by injecting uncertainty into its price dynamics, while speculators store in increasing numbers, with possibly a final "run" on the good. The stochastic price policies of a large number of such firms are shown to aggregate back to a price index growing at the rate of general inflation in response to which they arose. The model thus establishes that: (a) a constant rate of aggregate inflation can at the same time generate and cover up significant uncertainty and social costs at the microeconomic level; (b) speculation can be destabilizing and socially wasteful, even in the absence of shocks and imperfect information. Most importantly, they provide a theoretical foundation for the frequently encountered claim that inflation causes price uncertainty.

KEYWORDS: Inflation, speculation, price uncertainty, dynamic game.

INTRODUCTION

"Soon, nobody knew how much things cost any more. Prices were jumping up in a completely arbitrary manner; a box of matches cost, in a shop which had increased its prices at the right moment, twenty times as much as in another one, where a decent fellow was still selling his merchandise at the previous day's price. As a reward for his honesty, his shop was emptied within an hour, because the word was passed on, everyone rushed and bought what was for sale, whether they needed it or not" (Zweig (1943)).

THIS ACCOUNT OF THE Austrian hyperinflation of 1921–1922 dramatically illustrates two important aspects of inflationary economies: sellers face a crucial and repeated problem of when to adjust their prices, while buyers speculate on the timing of these increases to go on a buying-for-storage spree just before. Less extreme situations, such as oil shocks, the removal of subsidies, or even steady but high enough inflation also provide ample evidence that most goods can, and indeed will be stored if buyers expect their price to go up significantly (or shortages).[2]

The quotation also confirms that most prices are adjusted at discrete intervals and not continuously; Mussa (1981) found similarly that the frequency of price increases for selected commodities during the 1923 German hyperinflation was, in a sense, small in comparison to the speed of inflation. Indeed, changing prices entails some costs: new information must be gathered and disseminated, price

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[2] Goods which are not storable but for which intertemporal substitution of consumption is possible give rise to similar behavior.

tags, lists and catalogues updated, contracts and collusive agreements renegotiated, etc.; price changes may also trigger search by customers.

The optimal price policy for a monopolistic seller of a nonstorable good who faces a fixed cost of changing his price was characterized by Barro (1972) when the price must adapt to demand shocks, and by Sheshinski and Weiss (1977), (1979), (1983) when it must keep pace with environing inflation. The solution was shown to be an (S, s) rule, according to which (in the latter case) the real price is readjusted to some ceiling level S every time inflation has eroded it below some floor level s; with constant inflation, such adjustments occur with fixed periodicity. But storability endows the price adjustment problem with a new, speculative dimension, which generally renders such deterministic price policies suboptimal: if consumers knew that their supermarket or gas station adjusted its prices every Friday morning, they would store the day before, thereby depriving the seller of his sales at the peak and increasing them at the lowest point of the real price cycle. This prospect would in turn give him an incentive to advance the price increase to Thursday morning; buyers would then try to store on Wednesday instead, etc.

This paper solves the problem of the optimal price and storage strategies for a firm selling a storable good and its customers as the Markov perfect equilibrium of a dynamic game with infinite horizon. It shows in particular that the firm may find it profitable to inject randomness into its price dynamics, while, as time goes by before the price adjustment occurs, buyers store in increasing numbers, with possibly a final "run" on the good. The first result provides a theoretical foundation for the often-mentioned link between inflation and price uncertainty, and the second an accurate description of buyer behavior in inflationary situations, as well as a proof that speculation may be destabilizing, even in a context of perfect information.

The model and equilibrium conditions are presented in Section 1. Section 2 fully characterizes the possible types of equilibria, while Section 3 establishes existence and uniqueness. Section 4 relates the equilibrium to the no-storage case, and assesses the effects of inflation on price-storage dynamics, as well as welfare. Finally, Section 5 examines the cross-sectional distribution and aggregate price index of a large number of firms, and draws macroeconomic consequences from the results. Most proofs are gathered in appendices at the end of the paper. The main results are illustrated on Figures 1 to 4, in Section 2.4.

1. MODEL AND PRELIMINARY RESULTS

1.1. *Description of the Market*

THE FIRM: A monopolistic firm, selling a storable good[3] which lasts for two periods, faces an inflationary environment: its costs and all aggregate prices

[3] Unlike durable goods, which are not themselves consumed but yield a flow of services over time, storable goods disappear after consumption.

increase at a constant rate of π per period. With all nominal prices deflated by the aggregate price index, the firm operates with constant real costs, which are: a production cost of c per unit and a fixed cost of changing prices (so-called "menu cost") of β.[4] The firm is risk-neutral, infinitely-lived and maximizes in each period the expected present value (e.p.v.) of its profits, with a discount factor $\delta < 1$.

BUYERS: A unit continuum of infinitely lived, risk-neutral buyers (consumers, retailers, or other industries) maximize in each period the expected present value of their instantaneous utilities, with the same discount factor δ as the firm. Each of them requires one unit of the good per period, provided the real price is below a common reservation value $S > c$.[5] In every period, buyers consume any previous inventories, then buy from the firm to satisfy their current needs (if inventories are nonexistent or insufficient) as well as for storage until the next period, if they so desire. Storage is costly because of foregone interest on the value of goods stored, and of a constant real storage cost per unit.

The market's potential for speculation will be parameterized as follows: a fraction $1-x$ $(0 < x < 1)$ of buyers (hereafter referred to as speculators, or speculating customers) can store at a cost $\alpha < \delta S$, while the remaining x (nonspeculators) face a storage cost $\alpha' > \delta S$, which renders storage always unprofitable. Because the good is storable for one period only and consumers are satisfied with one unit (and assuming that transactions costs prohibit resales among different types) they never want to store more than one unit each; total storage is thus bounded by $1-x$.[6]

1.2. *The Game Between Firm and Customers*

TIMING OF MOVES: In every period, the firm decides whether to change its nominal price or let the previous one be eroded by inflation, then buyers make their purchasing and storage decisions; in the following period, the firm, having observed inventories, makes a new price decision; buyers then come back, etc.[7] This sequential timing and the continuum assumption on buyers correspond to what Gale (1982) terms a "repeated leader-follower" game. As mentioned previously, pure strategies will generally not be optimal; more likely, the firm will adjust its price at random intervals (or by random amounts) to deter speculation,

[4] Indexed contracts may arise when the firm sells to a few large customers, but are too costly to draw and enforce with many small buyers.

[5] Buyers have instantaneous utility $U(z, y) = y + S \cdot \min(z, 1)$, where z is consumption of the firm's good and y real income spent on others; their real income in each period is $I \geqslant S$. Their instantaneous indirect utility function (in the absence of storage) is thus $W(P, I) = I + \max(S - P, 0)$.

[6] One could also interpret $1-x$ as the total storage capacity of pure competitive speculators operating in a market where consumers cannot store themselves (e.g. Hart-Kreps (1986)); or alternatively, x as a flow of transient customers, renewed every period, or as the fraction of customers which the firm succeeds in rationing when they try to store. Finally, note that, because of risk-neutrality, there is no role for a future's market.

[7] Because individual buyers act nonstrategically, it makes in fact no difference whether the firm has observed previous storage when it sets the new price (alternating moves) or not (simultaneous moves).

while only a fraction of speculators will store (one unit each) in every period.[8]

ADJUSTMENT LEVEL: The most general strategy for the firm in each period is a probability distribution over $R_+ \times \{0,1\}$, i.e. real prices and the action of closing down—which brings the game to an end when future expected profits are negative. Bénabou (1986) treats the game in this general form, and shows that, as long as the cost of price adjustment is smaller than the maximum revenue from nonspeculating customers ($x(S-c) > \beta$), attention can be restricted to a *simpler game*, where in every period the firm only decides between adjusting its real price *back to the reservation level S* and not adjusting it.[9] For brevity, only the simplified version of the game is presented here, under the following assumption:

ASSUMPTION A: $x(S-c) > \beta$.

1.3. *Strategies and Equilibrium Concept*

Attention will be restricted to (stationary) Markov, or state-space, strategies, which, for each player, depend only on *payoff relevant* state variables, i.e. on those which directly affect the current and future flow payoffs from his decision.[10] The equilibrium concept is thus that of *Markov perfect equilibrium* (in short, MPE; cf. Maskin and Tirole (1987), (1988a), (1988b), Gertner (1986)). The first reason for this choice is to reduce the plethoric equilibrium set (folk theorem, e.g. Friedman (1987)) which arises when all history-dependent strategies are allowed.[11] The second one is a concern for robustness: because players react only to variables which constitute a physical intertemporal link in the game, MPE are more robust, both to renegotiation among players and to the specification of a finite or infinite horizon, than supergame-type equilibria based on threats of punishment for "immaterial" past behavior.[12]

[8] For an interpretation and formal justification of a mixed strategy equilibrium in a game of perfect information as the limit of pure strategy equilibria in the same game perturbed by an infinitesimal amount of incomplete information, cf. Harsanyi (1973) and Milgrom and Weber (1986).

[9] Because of the simplifying assumption made on preferences, the firm might want to "overshoot" to $S(1+\pi)$, so as to dry up inventories and force all buyers to buy at S in the next period. The presence of enough nonspeculators ($x(S-c) > \beta$) whose purchases would be lost for one period makes this (implausible) strategy less profitable than adjusting to S directly (cf. Bénabou (1986)). Under this condition, the unique (Markov perfect) equilibrium of the simplified game is also an equilibrium of the general one; it is the only (Markov perfect) equilibrium when $\alpha = c = 0$, and always the only one involving adjustment to a constant level; if other equilibria of the larger game exist, they must involve adjustments to (variable) real prices $S' \in [S, S/(1+\pi))$.

[10] If time itself is not payoff-relevant (as will be the case here), the state-space assumption implies stationarity of the strategies. Formally, a state variable z is payoff relevant for player j, whose action set, decision variable, and instantaneous payoff are Y_j, $y_j \in Y_j$, and $G^j(y_j, y_{-j}; z)$, if either Y_j depends on z, or for some (y_{1j}, y_{2j}), $y_{ij} \neq y_{2j}$, the function: $[z \to G^j(y_{1j}, y_{-j}; z) - G^j(y_{2j}, y_{-j}; z)]$ is not constant. In the differentiable case, this takes the form: $\partial^2 G^j / \partial y_j \partial z \neq 0$.

[11] A MPE is still a subgame perfect equilibrium when arbitrary history-dependent strategies are allowed; it is simply one where all players disregard payoff irrelevant variables (cf. Maskin and Tirole (1988a)).

[12] In a context of perfect information, and no multiple Nash equilibria in the single-period game.

In period n, denote by $P(n) \leqslant S$ the real price, and by $q_i'(n)$ and $q'(n) = (\int_0^1 q_i'(n)\, di)/(1-x)$ respectively the quantities stored by customer i and by speculators on average. Buyer i's payoff (already embodying the trivial resolution of his optimal consumption decision) is:

$$(1) \qquad G_i(n) = S - P(n)(1 - q_i'(n-1)) - q_i'(n)(P(n) + \alpha).$$

The first two terms represent current utility and the last one the cost of storing one unit. The only state variable which is payoff-relevant for the choice of $q_i'(n)$ is therefore $P(n)$. In particular, previously accumulated inventories $q_i'(n-1)$ are not payoff-relevant for the storage decision (although they are for the decision of how much to buy for current consumption). In the following period, the firm's payoff is:

$$(2) \qquad G_F(n+1) = (P(n+1) - c)[1 + (1-x)(q'(n+1) - q'(n))] \\ - \beta\Delta(P(n)/\theta, P(n+1)),$$

where $\theta \equiv 1 + \pi$ and $\Delta(y, z) \equiv 0$ if $y = z$, and 1 otherwise. This periods' storage $(1-x)q'(n+1)$ adds to sales, while last period's $(1-x)q'(n)$ subtracts from them. The only payoff-relevant state variables for the choice between $P(n+1) = S$ (adjustment) and $P(n+1) = P(n)/\theta$ (no adjustment) are thus $P(n)$ (because of the adjustment cost) and $q'(n)$ (inventories to be consumed at the expense of new sales). Markov strategies for this game are therefore defined as follows:[13]

For customer i: a mapping φ_i': $P \in [0, S] \to \varphi_i'(P) \in \{0,1\}$, specifying his storage decision as a function of the current real price.

For the firm: a mapping φ: $(P, q') \in [0, S] \times [0,1] \to \varphi(P, q') \in [0,1]$, specifying the probability with which the real price is adjusted (to S) as a function of its previous value and speculator's average inventory level q'.

1.4. *Dynamic Programming Problems*

Assume that an equilibrium exists, and denote by $\mathscr{W}_B(P, q_i')$ (resp. $\mathscr{V}_B(P, q', q_i')$) the maximized e.p.v. of customer i's utility when he and the other buyers are (resp. when the firm is) about to play, so that P represents the current (resp. previous) period's real price, and (q_i', q') the current inventory levels of customer i and of the average speculator. The functional equations associated to customer i's dynamic programming problem are, by (1):

$$(3a) \qquad \mathscr{W}_B(P, q_i') = S - P(1 - q_i') \\ + \max_{q_i'' \in \{0,1\}} \{-q_i''(P + \alpha) + \delta \mathscr{V}_B(P, \varphi'(P), q_i'')\},$$

$$(3b) \qquad \mathscr{V}_B(P, q', q_i') = \varphi(P, q') \cdot \mathscr{W}_B(S, q_i') + (1 - \varphi(P, q')) \cdot \mathscr{W}_B(P/\theta, q_i')$$

for all (P, q', q_i'). Note that customer i takes both $\varphi'(P)$ and $\varphi(P, q')$ as given. Define $\tilde{\varphi}(P) \equiv \varphi(P, \varphi'(P))$ for all P. Customer i's value function (at his

[13] Attention could even be restricted to $P \in \{S\theta^{-k} | k \in \mathbb{N}\}$; keeping $P \in [0, S]$ and dealing with functions instead of sequences is more convenient and allows the game to be solved for any initial price $P \leqslant S$.

decision nodes) $\mathscr{W}_B$ is therefore a solution to the Bellman equation:

$$\text{(3c)}\qquad \mathscr{W}_B(P,q_i') = \operatorname*{Max}_{q_i''\in\{0,1\}} \{S - P(1-q_i') - q_i''(P+\alpha) + \delta[\tilde{\varphi}(P)\cdot \mathscr{W}_B(S,q_i'') + (1-\tilde{\varphi}(P)) \cdot \mathscr{W}_B(P/\theta, q''_i)]\}$$

for all (P, q_i'), and his strategy φ_i' is an associated optimal control.

Similarly, let $\mathscr{W}_F(P,q')$ (resp. $\mathscr{V}_F(P,q')$) denote the maximized e.p.v. of the firm's profits when it is (resp. when buyers are) about to play, so that P represents the previous (resp. current) period's real price, and q' speculators' average current inventories. The functional equations associated to the firm's dynamic programming problem are, by (2):

$$\text{(4a)}\qquad \mathscr{W}_F(P,q') = \operatorname*{Max}_{q\in[0,1]} \{q(\mathscr{V}_F(S,q') - \beta) + (1-q)\mathscr{V}_F(P/\theta, q')\},$$

$$\text{(4b)}\qquad \mathscr{V}_F(P,q') = (P-c)[1 + (1-x)(\varphi'(P) - q')] + \delta \mathscr{W}_F(P, \varphi'(P))$$

for all (P, q'). The firm's value function (at its decision nodes) $\mathscr{W}_F$ is therefore a solution to the Bellman equation:

$$\text{(4c)}\qquad \mathscr{W}_F(P,q') = \operatorname*{Max}_{q\in[0,1]} \{q[-\beta + (S-c)(1 + (1-x)(\varphi'(S) - q')) + \delta \mathscr{W}_F(S, \varphi'(S))] + (1-q)[(P/\theta - c)(1 + (1-x)(\varphi'(P/\theta) - q')) + \delta \mathscr{W}_F(P/\theta, \varphi'(P/\theta))]\}$$

for all (P, q'), and its strategy φ is an associated optimal control.

1.5. *Normalizations*

The quantity state variables q' and q_i' will now be eliminated by a convenient normalization. By (3a) and (4b):

$$\text{(5)}\qquad \mathscr{W}_B(P,q_i') = \mathscr{W}_B(P,0) + Pq_i',$$

$$\text{(6)}\qquad \mathscr{V}_F(P,q') = \mathscr{V}_F(P,0) - (P-c)(1-x)q'$$

for all (P, q', q_i'). Define therefore the *normalized value functions* for a representative customer and for the firm as:

$$\text{(7)}\qquad \mathscr{W}(P) \equiv \mathscr{W}_B(P,0),$$

$$\text{(8)}\qquad \mathscr{V}(P) \equiv \mathscr{V}_F(P,0).$$

Both normalized value functions are measured from a decision node of buyers at which all have zero inventories. By (3a)–(3b), $\mathscr{W}$ satisfies:

$$\text{(9)}\qquad \mathscr{W}(P) = \operatorname*{Max}_{q_i''\in\{0,1\}} \{S - P - q_i''(P+\alpha) + \delta\tilde{\varphi}(P)\cdot[\mathscr{W}(S) + q_i''S] + \delta(1-\tilde{\varphi}(P)) \cdot [\mathscr{W}(P/\theta) + q''_i P/\theta]\}$$

which differs from (3c) by a constant only, and is therefore an equivalent way of defining speculator i's dynamic programming problem and determining his optimal strategy φ_i'. Similarly, by (4a)–(4b), $\mathscr{V}$ satisfies:

$$\begin{aligned}(10) \qquad \mathscr{V}(P) = \operatorname*{Max}_{q\in[0,1]} \{&(P-c)[1+(1-x)\varphi'(P)] \\ &+q\delta[\mathscr{V}(S)-\beta-(S-c)(1-x)\varphi'(P)] \\ &+(1-q)\delta[\mathscr{V}(P/\theta)-(P/\theta-c)(1-x)\varphi'(P)]\},\end{aligned}$$

and the corresponding set of optimal controls coincides with the set of optimal firm strategies $\tilde{\varphi}(P)=\varphi(P,\varphi'(P))$ on the equilibrium path (where q' is always equal to $\varphi'(P)$). Conversely, a function $\mathscr{V}$ solution to (10) is sufficient to completely define the firm's strategy and value *off* the equilibrium path as well. Define for all (P,q'): $\mathscr{V}_F(P,0)\equiv\mathscr{V}(P)$, $\mathscr{V}_F(P,q')$ as in (6), and $\mathscr{W}_F(P,q')$ as in (4a); it is easily verified that (10) implies that $\mathscr{W}_F$ is a solution to (4c), the Bellman equation for the firm's original dynamic programming problem. Moreover, if $q'=\varphi'(P)$, the right-hand sides of this equation and of (10) coincide (by construction); an optimal control $[P\to\tilde{\varphi}(P)]$ associated to $\mathscr{V}$ in (10) can therefore be extended into an optimal $[(P,q')\to\varphi(P,q')]$ associated to $\mathscr{W}_F$ in (4c), with $\varphi(P,\varphi'(P))=\tilde{\varphi}(P)$ for all P.

To summarize, attention can (and will henceforth) be restricted without loss of generality to *normalized values functions* and strategies defined over a *single state variable*:

DEFINITION 1.1: For any $t\in\mathbb{R}_+$, the market is said to be in *state t* if the real price charged by the firm is $P_t\equiv S(1+\pi)^{-t}=S\theta^{-t}$. Correspondingly, define: $V_t\equiv\mathscr{V}(P_t)$, $q_t\equiv\varphi(P_t,\varphi'(P_t))$, $p_t\equiv P_t-c$ for the firm, and: $W_t\equiv\mathscr{W}(P_t)$, $q_{it}'\equiv\varphi_i'(P_t)$, $q_t'\equiv\varphi'(P_t)$ for buyers.

If no adjustment takes place, a transition occurs to state $t+1$; after each adjustment, the game is back in state 0 and starts a new "cycle." On the equilibrium path, t will thus be the number of periods elapsed since the real price was last adjusted to S. The set of functions mapping $\mathbb{R}_+$ into $[0,1]$ will be denoted by $\mathscr{F}$. To minimize the new notation, $\mathscr{V},\mathscr{W},\varphi,\varphi_i'$ and φ' will now denote the elements of $\mathscr{F}$: $\mathscr{V}$: $t\to V_t$, $\mathscr{W}$: $t\to W_t$, φ: $t\to q_t$, φ_i': $t\to q_{it}'$, φ': $t\to q_t'$. For all $y\in\mathbb{R}$, Int$[y]$ will denote its integer part, and $K[y]\equiv -\mathrm{Int}[-y] = \min\{k\in\mathbb{N}\,|\,k\geqslant y\}$.

The (normalized) Bellman equation (9) for a buyer's problem becomes:

$$\begin{aligned}(11) \qquad W_t = \max_{q_i'\in\{0,1\}} \{&S-P_t-q_i'(P_t+\alpha)+\delta[q_t(W_0+q_i'P_0) \\ &+(1-q_t)(W_{t+1}+q_i'P_{t+1})]\}.\end{aligned}$$

If the firm raises (resp. does not raise) its price in the next period, buyer i will obtain a value equal to W_0 (resp. W_{t+1}) plus savings of P_0 (resp. P_{t+1}) if he had stored.

Similarly, the (normalized) Bellman equation (10) for the firm's problem becomes:

$$(12)\qquad V_t = \max_{q\in[0,1]}\left\{p_t(1+(1-x)q_t') + \delta\left[q(V_0-\beta-(1-x)q_t'p_0) + (1-q)(V_{t+1}-(1-x)q_t'p_{t+1})\right]\right\}.$$

The firm randomizes between states zero and $t+1$, which yield values respectively equal to $V_0-\beta$ and V_{t+1}, minus the profits at the new price (resp. P_0 or P_{t+1}) on the sales $(1-x)q_t'$ lost because of storage.

The operators on $\mathscr{F}$ associating to any functions $\mathscr{W}$: $t\to W_t$ and $\mathscr{V}$: $t\to V_t$ the right-hand sides of (11) and (12) respectively, trivially verify Blackwell's (1965) sufficient conditions (monotonicity and discounting) guaranteeing that they are contractions. Therefore, for any $\varphi\in\mathscr{F}$, (11) has a unique solution $\mathscr{W}\in\mathscr{F}$, and a nonempty set of optimal controls denoted $\mathscr{R}'(\varphi)$; by linearity, $\varphi_i'\in\mathscr{R}'(\varphi)$ for almost all $i\in[0,1]$ if and only if $\varphi'\equiv\int_0^1\varphi_i'\,di\in\mathscr{R}'(\varphi)$. Similarly, for any $\varphi'\in\mathscr{F}$, (12) has a unique solution $\mathscr{V}$ and a nonempty set of optimal controls denoted $\mathscr{R}(\varphi')$. A MPE is a fixed point of the correspondence on $\mathscr{F}\times\mathscr{F}$:

$$(\varphi,\varphi')\to\mathscr{R}(\varphi')\times\mathscr{R}'(\varphi).$$

Usual existence theorems are not applicable to this problem. The proof of existence is therefore constructive, and also gives uniqueness and a full characterization of the equilibrium; it proceeds in two stages.

1.6. *Equilibrium and Continuation Value Equilibrium*

In the first stage (Section 2), the endogenous value V_0 appearing in the right hand side of (12) is treated as a parameter $V\in\mathbb{R}_+$. This amounts to replacing the original game by one which terminates as soon as the firm adjusts its real price, at which time it receives an exogenous continuation value V, but must buy back all inventories at the real price S. The Bellman equation (12) for the firm's problem then becomes:

$$(12')\qquad V_t = \max_{q\in[0,1]}\left\{p_t(1+(1-x)q_t') + \delta\left[q(V-\beta-(1-x)q_t'p_0) + (1-q)(V_{t+1}-(1-x)q_t'p_{t+1})\right]\right\}.$$

Blackwell's theorem also applies to (12'), which therefore has a unique solution and a nonempty set of optimal controls, for any $\varphi'\in\mathscr{F}$. But in contrast to the original game which is cyclical, the continuation value game can be shown to end within an (endogenous) horizon $\overline{T}(V)<+\infty$. It can then be solved from $\overline{T}(V)$ backwards, establishing the existence for every V of a unique (Markov) *continuation value equilibrium* (φ_V,φ_V',V), which can be fully characterized.

DEFINITION 1.2: A (Markov) continuation value equilibrium (CVE) is a triplet $(\varphi,\varphi',V)\in\mathscr{F}\times\mathscr{F}\times\mathbb{R}_+$ such that (φ,φ') are optimal controls in the Bellman equations (11) and (12') respectively.

In the second stage of the construction (Section 3), equilibria of the original game are derived as fixed points. In a CVE $(\varphi_V, \varphi'_V, V)$, the firm's e.p.v. of profits in state zero, i.e. V_0, can be computed recursively from (12′) as a function $f(V)$. The CVE is an equilibrium of the full game if and only if (12′) coincides with the original (12), i.e. if $V = V_0 = f(V)$.

The idea thus consists in replacing an infinite-horizon game by a parametrized family of finite horizon ones, and a fixed-point problem in the functional space $\mathscr{F} \times \mathscr{F}$ by one in $\mathbb{R}_+$.

The remainder of this section, and Section 2, therefore deal not only with MPE but more generally with CVE. The necessary and sufficient conditions for the linear maximization problems (11) and (12′) are:

Buyers:

$$(13) \quad \begin{cases} \text{if } P_t + \alpha > \delta(q_t P_0 + (1-q_t)P_{t+1}), & \text{then } q'_{it} = 0\ (\forall\, i), \text{ or: } \ q'_t = 0; \\ \text{if } P_t + \alpha < \delta(q_t P_0 + (1-q_t)P_{t+1}), & \text{then } q'_{it} = 1\ (\forall\, i), \text{ or: } q'_t = 1; \\ \text{if } P_t + \alpha = \delta(q_t P_0 + (1-q_t)P_{t+1}), & \text{then } q'_{it} \in \{0,1\}\ (\forall\, i), \\ & \text{or: } \ q'_t \in [0,1]. \end{cases}$$

Speculators simply compare the cost of buying and storing an extra unit with the discounted value of the price expected to prevail in the following period. In case of equality, each of them is indifferent between storing and not storing, so any proportion may decide to do so.

Firm:

$$(14) \quad \begin{cases} \text{if } q'_t(1-x)(p_0 - p_{t+1}) < V - \beta - V_{t+1}, \text{ then } q_t = 1; \\ \text{if } q'_t(1-x)(p_0 - p_{t+1}) > V - \beta - V_{t+1}, \text{ then } q_t = 0; \\ \text{if } q'_t(1-x)(p_0 - p_{t+1}) = V - \beta - V_{t+1}, \text{ then } q_t \in [0,1]. \end{cases}$$

The firm thus compares the net valuations $V - \beta - (1-x)q'_t p_0$ and $V_{t+1} - (1-x)q'_t p_{t+1}$ corresponding to the two possible transition states; or equivalently, the increment in normalized valuation $V - \beta - V_{t+1}$ resulting from adjustment with the corresponding increment in expected lost revenues (p_0 is lost rather than p_{t+1}, on $q'_t(1-x)$ customers). In case of equality, it is indifferent and can randomize its decision.

1.7. *Critical States for Deterministic Price Adjustments*

Two critical states which play an important role in buyers' and the firm's equilibrium strategies will now be introduced; they arise naturally when one examines under which conditions a deterministic price adjustment—defined as a jump: $q_{t-1} = 0$ and $q_t = 1$ ($q_{-1} \equiv 0$)—is or is not possible.

Let $q_t = 1$; the condition required by (13) for buyers' indifference in state t becomes: $P_t + \alpha = \delta P_0$ or $\theta^{-t} = \delta - \alpha/S$. Since $\alpha < \delta S$, define:

$$(15) \qquad \tau = \frac{\log[1/(\delta - \alpha/S)]}{\log(\theta)}.$$

When faced with a sure price adjustment in the following period, all speculators store if $t > \tau$, none do if $t < \tau$, and they are indifferent if $t = \tau$. Indeed, only after the real price has fallen below P_τ do the savings realized by storing justify the necessary costs. For the x customers with storage cost $\alpha' \geqslant \delta S$ (nonspeculators), speculation is never profitable ($\tau' \equiv +\infty$); if $x = 1$, the game thus reduces to the optimization by the firm of the frequency of price adjustments (Sheshinski and Weiss (1977)).

Let now $q_t = 1$ for $t > \tau$, so $q_t' = 1$, inflicting on the firm a sure loss of $(1 - x)(\delta p_0 - p_t)$. If it is too large, the firm will try to avoid reaching state t by implementing the price increase earlier with positive probability ($q_{t-1} > 0$, if $t \geqslant 1$), thereby precluding a deterministic adjustment in state t. Since $\delta p_0 - p_t$ increases with t, intuition suggests the existence of a critical state μ such that deterministic price adjustments with storage by all speculators cannot be optimal for the firm in states $t > \mu$. The following theorem indeed establishes that:

$$\mu = \frac{\log\left[(1 - z/\theta)/(1 - z)\right]}{\log(\theta)}, \qquad \text{where } z = \frac{1}{(2 - x)}. \tag{16}$$

THEOREM 1: *There exists* $\tau(\alpha/S, \delta, \pi) > 0$ *and* $\mu(x, \pi) \in (0, 1/(1 - x))$ *such that, in any continuation value equilibrium*: (*i*) *For all* $t < \tau$, $q_t' = 0$. (*ii*) *For all* $t > \tau$, *if* $q_t = 1$ *then* $q_t' = 1$. (*iii*) *For all* $t \geqslant 1$, *if* $t > \max(\tau, \mu)$ *and* $q_t = 1$ *then* $q_{t-1} > 0$. (*iv*) $\partial\tau/\partial\alpha > 0$; $\partial\tau/\partial\delta < 0$; $\partial\tau/\partial\pi < 0$; $\partial\mu/\partial x > 0$; $\partial\mu/\partial\pi < 0$.

PROOF: Cf. Appendix 1.

The most important result is that there can be no deterministic price adjustment in a state $t > \max(\tau, \mu)$. Both τ and μ are independent of the continuation value game played, i.e. of V. The generic conditions $\tau \notin \mathbb{N}$ and $\mu \notin \mathbb{N}$, or $\text{int}[\tau] < \tau$ and $\text{int}[\mu] < \mu$, will be assumed from here on.

2. CHARACTERIZATION OF THE EQUILIBRIUM

The existence of a unique CVE for any continuation value V in an appropriate interval will now be established. A CVE, and in particular any full equilibrium of the game, is shown to generally consist of *three phases*, separated by thresholds $\underline{T}$ and $\overline{T}$: pure strategies during $[0, \underline{T})$, then mixed during $[\underline{T}, \overline{T})$, then again pure strategies during $[\overline{T}, +\infty)$.

The construction proceeds in three steps. First, the phases of pure strategies are examined; it is shown that there exist T^* and $\overline{T}$ such that $q_t = 0$ for $t < \min(T^*, \tau - 1)$ and $q_t = 1$ for $t > \overline{T}$. The intermediate phase of mixed strategies is then analyzed; q_t is shown to be a computable function Q_t of the state t, derived from (13), while q_t' is obtained as the solution $Q_{t,V}'$ to a linear difference equation with variable (state-dependent) coefficients, derived from (14) and (12′). Finally, this system is solved backwards from $\overline{T}$ to determine $\underline{T} \leqslant T^*$.

2.1. *The Phases of Pure Strategies*

Assume first that buyers never store. The opportunity cost of postponing the price adjustment for one period is then $(1-\delta)(V-\beta)$, while the gain from that postponement is next period's net real revenue p_{t+1}. The firm therefore adjusts its price, with probability one, in the state T^* such that these two quantities are equal;[14] it thus follows an (S,s) rule, with $s-c=p_{T^*+1}=(1-\delta)(V-\beta)$ (as in Sheshinski and Weiss (1977)). Similarly, if the $1-x$ speculators always store, the optimal policy is an (S,s) rule, with $s-c$ equated to the opportunity cost $(1-\delta)(V-\beta-(1-x)p_0)$.

DEFINITION 2.1: Let $\Gamma=[(p_0-\delta\beta)/(1-\delta),\, p_0/(1-\delta)]$. For all $V\in\Gamma$, define $T^*(V)$ and $\overline{T}(V)$ by:

$$(17)\qquad p_{T^*(V)+1}=(1-\delta)(V-\beta);\quad p_{\overline{T}(V)+1}=(1-\delta)(V-\beta-(1-x)p_0).$$

For all $V\in\Gamma$: $(1-\delta)(V-\beta-(1-x)p_0)\geqslant p_0[1-(1-\delta)(1-x)]-\beta>xp_0-\beta>0$ by Assumption (A); hence $0\leqslant T^*+1<\overline{T}+1<+\infty$ (the dependence on V will be omitted when no confusion arises). The states T^* and $\overline{T}$, which determine adjustment in the benchmark cases where speculators never or always store, also provide bounds on the actual time of adjustment:

LEMMA 2.1: *Let (φ,φ',V) be a continuation value equilibrium, with $V\in\Gamma$. For all $t\in\mathbb{R}_+$: (i) If $t<\min(T^*,\tau-1)$, then $q_t=0$. (ii) If $t>\overline{T}$ or $t=\overline{T}\leqslant\tau$, then $q_t=1$. (iii) If $t=\overline{T}>\tau$, then $q_t\geqslant(P_t+\alpha-\delta P_{t+1})/[\delta(P_0-P_{t+1})]$ and $q_t'=1$.*

PROOF: Cf. Appendix 2.

Thus, the game starts with a phase of pure, inactive, strategies ($q_t=q_t'=0$ for $t<\min(T^*,\tau-1)$) and ends with a phase of pure, active strategies ($q_t=q_t'=1$ for $t>\max(\overline{T},\tau)$). In particular, if the price has not been adjusted by $\overline{T}$, the adjustment takes place with certainty in the following period, even though all speculators have stored: the firm cannot wait any longer and gives up its attempts at a surprise adjustment.

[14] Since the firm plays at discrete intervals of time, it will in fact adjust the price when the state $K[T^*(V)]=\min\{k\in\mathbb{N}\mid k\geqslant T^*(V)\}$ is reached.

2.2. *The Phase of Mixed Strategies*

By (13), the probability q_t of a price increase in the next period which leaves speculators indifferent between storing at the price P_t ($t > \tau$) and not storing is:

$$(18) \qquad q_t = \frac{P_t + \alpha - \delta P_{t+1}}{\delta(P_0 - P_{t+1})} = \frac{\theta^{-t}(1/\delta - 1/\theta) + \alpha/\delta S}{1 - \theta^{-(t+1)}}$$

which is less than one, since $t > \tau$, or $P_t + \alpha < \delta P_0$. Similarly, the fraction q_t' of speculating customers storing in state t which leaves the firm indifferent between the real prices P_0 and P_{t+1} in the next period is, by (14): $q_t' = (V - \beta - V_{t+1})/[(1-x)(p_0 - p_{t+1})]$. But by (12′):

$$(19) \qquad V_{t+1} \geqslant p_{t+1}(1 + (1-x)q_{t+1}') + \delta(V - \beta - (1-x)q_{t+1}' p_0),$$

with equality if $q_{t+1} > 0$. During a mixed strategy phase, q_t' therefore obeys the difference equation:

$$(20) \qquad q_t' = \frac{\delta p_0 - p_{t+1}}{p_0 - p_{t+1}} \cdot q_{t+1}' + \frac{(1-\delta)(V-\beta) - p_{t+1}}{(1-x)(p_0 - p_{t+1})}.$$

Definition 2.2: Define the following functions:

$$\text{(i)} \qquad Q\colon t \in \mathbb{R}_+ \to Q_t = \min\left\{1, \frac{\theta^{-t}(1/\delta - 1/\theta) + \alpha/\delta S}{1 - \theta^{-(t+1)}}\right\}.$$

(ii) For all $V \in \Gamma$, ψ_V: $(t, y) \in [\tau - 1, +\infty) \times \mathbb{R} \to \psi_{t,V}(y)$ given by:

$$\psi_{t,V}(y) = \frac{\delta p_0 - p_{t+1}}{p_0 - p_{t+1}} \cdot y + \frac{(1-\delta)(V-\beta) - p_{t+1}}{(1-x)(p_0 - p_{t+1})} \equiv a_{t+1} \cdot y + b_{t+1,V}.$$

During a phase of mixed strategies, $q_t = Q_t$ and $q_t' = \psi_{t,V}(q_{t+1}')$ by (18) and (20). Since $q_t' = 1$ for $t > \max(\overline{T}(V), \tau)$ by Lemma 2.1 and Theorem 1, q_t' must coincide, *during the mixed strategy phase*, with the solution $Q_{t,V}'$ to the difference equation:

$$(21) \qquad Q_{t,V}' = a_{t+1} \cdot Q_{t+1,V}' + b_{t+1,V} = \psi_{t,V}(Q_{t+1,V}')$$

for the same terminal condition at max $(\overline{T}(V), \tau)$. This function is constructed below on $[\tau - 1, \overline{T}(V)]$ (when nonempty) by backwards induction.

Definition 2.3: For all $V \in \Gamma$, define $Q_{t,V}'$: $t \in [\tau - 1, +\infty) \to Q_{t,V}'$ by:[15]

$$\text{(i)} \qquad Q_{t,V}' = 1 \quad \text{on} \quad [\max(\tau - 1, \overline{T}(V)), +\infty);$$

$$\text{(ii)} \qquad Q_{t,V}' = \psi_{t,V}(Q_{t+1,V}') = \psi_{t,V} \circ \cdots \circ \psi_{t+k,V}(1) \quad \text{on} \quad [\tau - 1, \overline{T}(V)),$$

where $k \equiv \min\{j \in \mathbb{N} \mid t + j + 1 \geqslant \overline{T}\}$.

[15] Throughout the paper $[a, b) = (a, b) \equiv \varnothing$ for all (a, b) with $a > b$.

Unlike Q_t, $Q'_{t,V}$ depends on V; this subscript will be omitted when no confusion results. The following lemma describes the dynamics of Q and Q', which will be shown below to define the equilibrium strategies ($q_t = Q_t$, $q'_t = Q'_t$) during a unique, connected, phase of mixed strategies.

LEMMA 2.2: *(i) The function* Q: $t \to Q_t$ *is continuous, equal to* 1 *for* $t \leqslant \tau$, *and then decreasing to its limit* $Q_\infty = \alpha/\delta S \geqslant 0$. *(ii) For all* $V \in \Gamma$ *such that* $\overline{T}(V) > \tau - 1$, *the function* Q'_V: $t \to Q'_{t,V}$ *is continuous and increasing on* $[\tau - 1, \overline{T}(V)]$.

PROOF: Cf. Appendix 2.

INTERPRETATION: During the phase of mixed strategies the (conditional) probability Q_t of a price adjustment in state t is decreasing. This somewhat surprising result can be explained as follows. Not only does the real gain $\delta P_0 - P_t - \alpha$ realized by storing before a price increase become larger over time, but the loss $P_t + \alpha - \delta P_{t+1}$ incurred if the adjustment does not materialize becomes smaller; to deter speculation—or at least keep speculators indifferent between storing and not storing—the probability of realizing the gain must decrease over time.[16]

The increasing fraction $(1-x)Q'_t$ of buyers who store during the phase of mixed strategies accords well with observation. It is not due to storage's increasing profitability, but again to the necessity of keeping the firm indifferent. Both the benefit from price adjustment $V_0 - \beta - V_{t+1}$ and the corresponding loss per storing customer $p_0 - p_{t+1}$ increase with t, but the former faster than the latter; to achieve indifference on part of the firm, the number $(1-x)Q'_t$ of storing customers must be increasing.

2.3. *Characterization, Existence, and Uniqueness of a CVE*

Since $Q'_t \in (0,1)$ leads to indifference for the firm, $\psi_t(q'_{t+1})Q'_t < 0$ means that not adjusting the price is strictly preferred in state t (similarly, $\psi_t(1) > 1$ means that adjustment is strictly preferred for $t > \overline{T}$). The time $\underline{T}$ at which the firm starts adjusting with positive probability can therefore be computed by moving backwards from $\overline{T}$, and looking for the unique zero (if any) of the continuous and decreasing (as t decreases) function Q': $t \to Q'_t$.

DEFINITION 2.4: For all $V \in \Gamma$, define $\underline{T}(V)$ as follows: (i) If $T^*(V) \leqslant \tau - 1$, $\underline{T}(V) = T^*(V)$. (ii) If $T^*(V) > \tau - 1$, $\underline{T}(V) = \min\{t \in [\tau - 1, \overline{T}(V)] | Q'_{t,V} \geqslant 0\}$.

[16] While this result may not remain when customers are sufficiently heterogeneous (with for instance a continuous distribution of storage costs) it is an important warning against the fallacious intuition that the price increase should always grow more and more likely.

Note that $0 \leqslant \underline{T}+1 < \overline{T}+1$ (the argument V will be omitted when no confusion results). The equilibrium configuration will be determined in particular by the relative positions of $\underline{T}$ and $\overline{T}$ (beginning and end of the phase during which the price is adjusted with positive probability in the continuation value game corresponding to V), with respect to the fixed τ and μ (minimum date for storage to start, and maximum date until which a deterministic adjustment can be optimal if all speculators have stored). It is fully characterized by the following definition, which pieces together, at $\underline{T}$ and $\overline{T}$, the *three phases of the equilibrium*:

DEFINITION 2.5: Define on Γ the correspondences Ω_F, Ω_B, and Ω as follows:

(i) $$\Omega_F(V) = \{\varphi \in \mathscr{F} | \forall\, t\colon q_t = 0 \text{ if } t \in [0, \underline{T});\ q_t \in [0, Q_t] \text{ if } t = \underline{T};\ q_t = Q_t \text{ if } t \in (\underline{T}, \overline{T});\ q_t \in [Q_t, 1] \text{ if } t = \overline{T};\ q_t = 1 \text{ if } t \in (\overline{T}, +\infty)\}.$$

(ii) $$\Omega_B(V) = \{\varphi' \in \mathscr{F} | \forall\, t\colon q'_t = 0 \text{ if } t \in [0, \max(\tau, \underline{T}));\ q'_t = 0 \text{ if } t = \underline{T} \geqslant \tau;\ q'_t \in [0, Q'_t] \text{ if } t = \tau > \underline{T};\ q'_t = Q'_t \text{ if } t \in (\max(\tau, \underline{T}), +\infty)\}.$$

(iii) $$Q(V) = \begin{cases} \Omega_0(V) \equiv \Omega_F(V) \times \Omega_B(V) \\ \qquad \text{unless } \underline{T} = \tau - 1 \geqslant 0 \text{ and } \psi_{\tau-1}(q'_\tau) \neq 0; \\ \Omega_1(V) \equiv \Omega_0(V) \cap \{\varphi | q_{\underline{T}} = Q_{\underline{T}}\} \\ \qquad \text{if } \underline{T} = \tau - 1 \geqslant 0 \text{ and } \psi_{\tau-1}(q'_\tau) > 0; \\ \Omega_2(V) \equiv \Omega_0(V) \cap \{\varphi | q_{\underline{T}} = 0\} \\ \qquad \text{if } \underline{T} = \tau - 1 \geqslant 0 \text{ and } \psi_{\tau-1}(q'_\tau) < 0. \end{cases}$$

The correspondence Ω_F uniquely determines the firm's strategy q_t in all states $t \in \mathbb{R}_+$ except possibly $\underline{T}$ and $\overline{T}$, while Ω_B uniquely determines speculators' (average) strategy q'_t in all states except possibly τ. Uniqueness is from here on to be understood as uniqueness up to these possible indeterminacies. The following result is central to the paper.

THEOREM 2.1: *For any $V \in \Gamma$, there exists a unique continuation value equilibrium $(\varphi_V, \varphi'_V, V)$, and the strategy pair (φ_V, φ'_V) is uniquely determined as the solution to: $(\varphi, \varphi') \in \Omega(V)$. Moreover $\underline{T}(V) \leqslant T^*(V)$, with strict inequality if and only if $T^*(V) > \tau - 1$.*

PROOF: It is based on the simultaneous resolution of (12′)–(13)–(14) by backwards induction, from $\overline{T}$ to zero; cf. Appendix 2.

The first part of the theorem and the definition of $\Omega(V)$ confirm that equilibrium strategies are first equal to zero, then mixed according to Q_t for the firm and Q_t' for speculators, then equal to one. The second part is quite intuitive: if $T^* + 1 \leqslant \tau$, the firm can adjust its price at $T^* + 1$ before speculation becomes profitable ($\underline{T} + 1 = T^* + 1$); if $T^* + 1 > \tau$, on the contrary, speculators try to store before the price increase, while the firm tries to adjust its price before too many of them store; as in many models of asset price determination (e.g. foreign exchange), this causes part of the price changes (here, the total change but with probability less than one) to take place *earlier* than it would in the absence of speculation ($\underline{T} + 1 < T^* + 1$). In addition to advancing the potential price increase, the firm randomizes it, when $\tau < \underline{T} < \overline{T}$; but since both sides play at discrete intervals, mixed strategies are effectively implemented on the equilibrium path only if $[\underline{T}, \overline{T})$ contains an integer, or $K[\underline{T}(V)] < K[\overline{T}(V)]$ (recall that $K[y] \equiv \min\{k \in \mathbb{N} | k \geqslant y|, \forall y \in \mathbb{R}\}$). The following result gives a sufficient[17] condition for this to occur:

PROPOSITION 2.2: *If* $\max(\tau, \mu) \leqslant K[\overline{T}(V)]$, *then* $K[\underline{T}(V)] < K[\overline{T}(V)]$, *so that the equilibrium outcome is stochastic.*

PROOF: Cf. Appendix 2.

2.4. *The Four Possible Forms of Equilibrium*

Definition 2.5 completely describes continuation value equilibria,[18]—and in particular any full equilibrium—of the game: depending on the relative positions of I, $\overline{T}$, τ, and μ, they can take one of the *four basic forms* illustrated in Figures 1 to 4, on which the solid dots indicate the discrete states in which players may act.[19]

A. Pure Strategy Equilibrium: When $\overline{T} \leqslant \tau$, the equilibrium involves only pure strategies: the firm increases its price if the time elapsed since the last adjustment is greater than $\underline{T}$, and all speculators store if it is greater than τ. The firm in fact adopts an (S, s) rule, resulting in price adjustments of periodicity $K[\underline{T}] + 1$.

A.1: When $K[\underline{T}] \leqslant \text{int}[\tau]$, this adjustment occurs without any storage; it is the discrete time analog of the solution of Sheshinski and Weiss (1977), who deal with the limiting case $\tau = +\infty$ (hence $\underline{T} = T^* = \overline{T}$); cf. Figure 1.

A.2: When $K[\underline{T}] = \text{int}[\tau] + 1$ (which requires $K[\overline{T}(V)] \leqslant \text{int}[\mu]$), the adjustment occurs with all speculators storing; this case is identical, for all practical purposes, to B.2 below.

[17] The condition is almost necessary as well: when $\tau \leqslant K[\overline{T}] < \mu$, $0 < \overline{T} - \underline{T} < 1$, but $[\underline{T}, \overline{T})$ may still happen to contain an integer.

[18] For all $V \in \Gamma$; it is easily shown, from (12) and (13), that in any CVE V_0 must belong to Γ.

[19] The full equilibrium of the game can indeed take any of the four basic forms identified here for continuation value equilibria, depending on parameter values: cf. Table I in Section 4 and footnote (25) below.

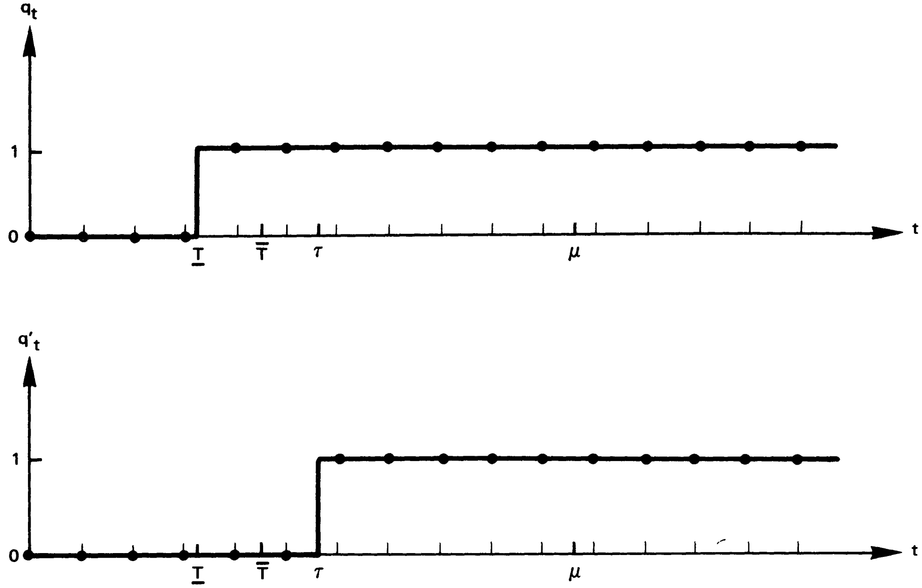

FIGURE 1.—Pure strategy equilibrium with no storage.

B. Mixed Strategy Equilibrium: When $\overline{T} > \tau$, there is a nonempty phase $(\max(\tau, \underline{T}), \overline{T})$ of mixed strategies on both sides; given that players act at discrete intervals, three types of outcomes are possible.

B.1. Mixed strategy equilibrium with deterministic outcome and no storage: When $K[\underline{T}] \leqslant \text{int}[\tau]$, the outcome is again an adjustment of periodicity $K[\underline{T}] + 1 \leqslant \text{int}[\tau] + 1$ and no storage. Only if the firm deviated, so that real price dropped below P_τ, would mixed strategies be implemented (cf. Figure 2); the ensuing outcome would then be similar to B.3 below.

B.2. Mixed strategy equilibrium with deterministic outcome and full storage: When $K[\underline{T}] > \text{int}[\tau]$ and the interval $(\underline{T}, \overline{T})$ contains no integer (which requires that $K[\overline{T}] \leqslant \text{int}[\mu] < 1/(1-x)$), the phase of mixed strategies is so short that players' actual moves "skip over it" to the final pure strategy phase, and the outcome consists of price adjustments of periodicity $K[\underline{T}] + 1$, with storage by all $1 - x$ speculators. This case occurs when $1 - x$ is small (so that μ is large): the firm maintains a deterministic (S, s) rule, forfeiting the small loss from storage for the benefit of charging the maximum price to nonspeculators (cf. Figure 3).

B.3. Mixed strategy equilibrium with stochastic outcome and increasing storage: The case where $K[\underline{T}] > \text{int}[\tau]$ and the interval $[\underline{T}, \overline{T})$ contains at least one integer (which always occurs when $K[\overline{T}] > \max(\text{int}[\tau], \text{int}[\mu])$), gives rise to a radically new type of outcome. The firm might be said to follow an $(S, \tilde{s})$ real price rule, where the tilde indicates a *random* variable, which here has support in

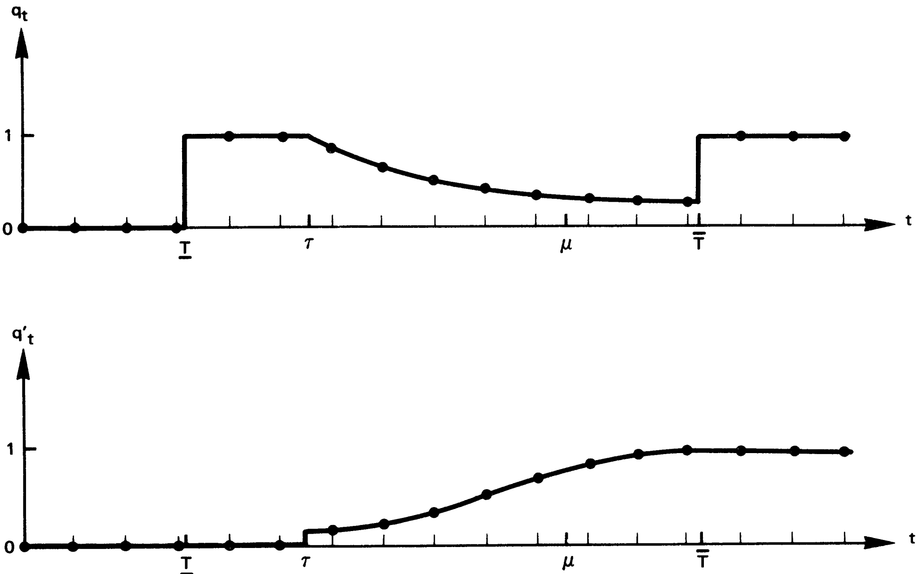

FIGURE 2.—Mixed strategy equilibrium, deterministic outcome with no storage.

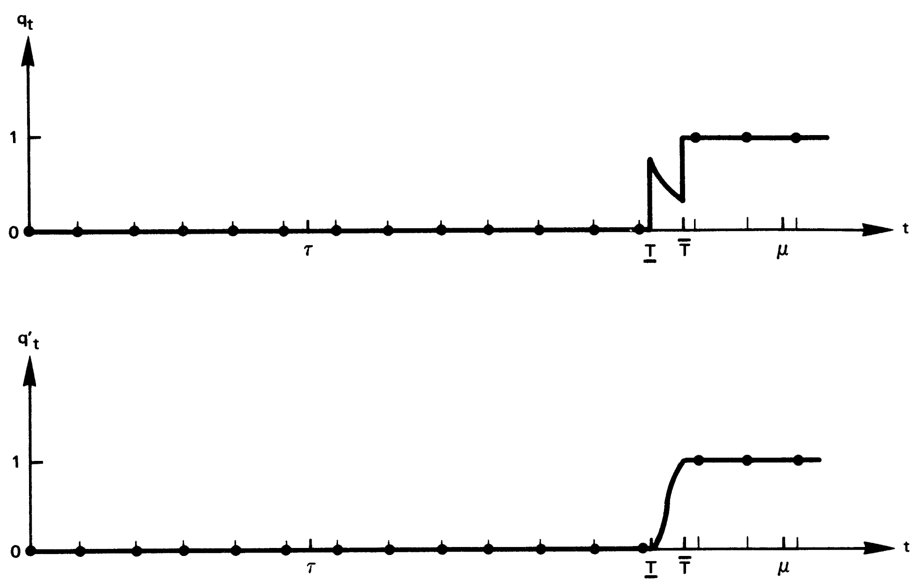

FIGURE 3.—Mixed strategy equilibrium, deterministic outcome with full storage.

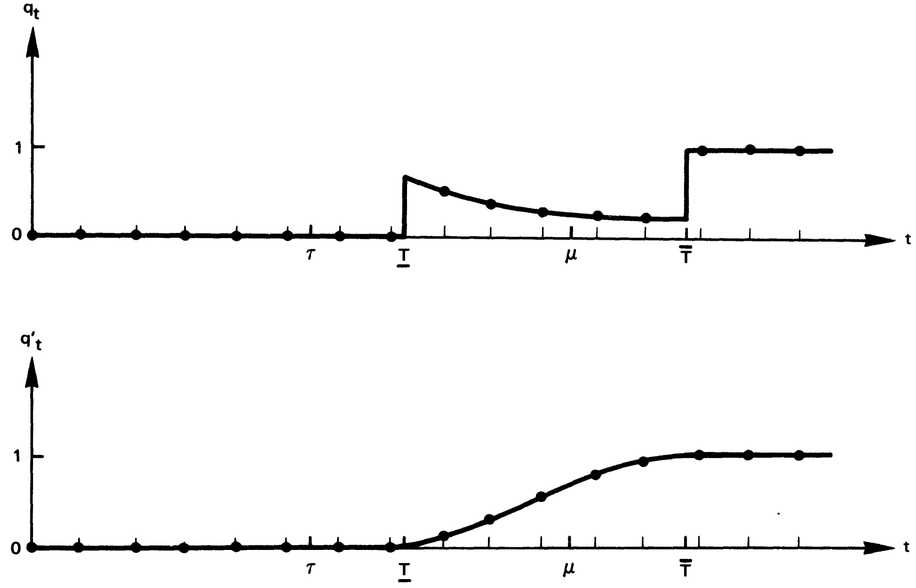

FIGURE 4.—Mixed strategy equilibrium, stochastic outcome with increasing storage.

$[P_{\underline{T}+1}, P_{\bar{T}+1}]$.[20] The nominal price remains pegged for $K[\underline{T}]$ periods; in every following period there is a probability Q_t that a price increase is about to take place; if it still has not occurred after $K[\bar{T}]$ periods, it then takes place with probability one in the next period. As to buyers, they store in increasing numbers $(1-x)Q_t'$ until the adjustment takes place; if it has not occurred after $K[\bar{T}]$ periods, there is a generalized "run" on the good by speculators (cf. Figure 4).

The discontinuous profile of q_t can be intuitively understood as follows. Absent storage, the firm would adjust with probability one at T^*, resulting in a jump of size one in q_t, as in Figure 1; in response to speculation, it spreads out the probability mass on both sides of T^*, i.e. on $[\underline{T}, \bar{T}]$, with parts of the original discontinuity remaining at the extremities. In earlier states $[\underline{T}, T^*]$, the firm tries to take advantage of speculators' low inventories (cf. Theorem 2.1); in later states $[T^*, \bar{T}]$, high inventories reduce the profitability of adjustment, making the firm less eager to adjust (cf. Section 2.1).

3. EXISTENCE AND UNIQUENESS OF THE EQUILIBRIUM

The preceding section's results on continuation value equilibria will now be used to establish the existence and uniqueness of a full (Markov perfect) equilibrium of the original game.

[20] With a more general (elastic) demand, the optimal policy would involve randomization over the upper bound as well $((\tilde{S}, \tilde{s})$ policy).

3.1. *Existence*

Note first (cf. Section 1.6) that if (φ, φ') is an equilibrium, then (φ, φ', V_0) is trivially a CVE, where V_0 is the value at zero of the solution to (12). Conversely, to every V corresponds (Theorem 2.1) a unique CVE $(\varphi_V, \varphi'_V, V)$;[18] φ'_V and φ_V are then optimal controls in (11) and (12′), which coincide with the original Bellman equations (11)–(12) if and only if the value V_0 at zero of the solution to (12′) is equal to V. This e.p.v. of profits in state zero resulting from (φ_V, φ'_V) can be computed as a function $V_0 = f(V)$; a MPE is therefore uniquely defined by a fixed point of the function f, which will now be specified.

For all $V \in \Gamma$ and any $(\varphi, \varphi') \in \Omega(V)$, define from here on $\underline{K} = K[\underline{T}]$, $K^* = K[T^*]$, and $\overline{K} = K[\overline{T}]$. By Definition 2.5, $q_k = q'_k = 0$ for $k < \underline{T}$, while two cases are possible in state $\underline{K} \geqslant \underline{T}$: either $\underline{T} \notin \mathbb{N}$, so $\underline{K} > \underline{T}$ and $q_{\underline{K}} = Q_{\underline{K}} > 0$, or $\underline{K} = \underline{T} \in \mathbb{N}$, hence $\underline{T} \neq \tau - 1$, so $q_{\underline{K}} \in [0, Q_{\underline{K}}] \neq \{0\}$. In both cases, adjusting the price is one of the firm's preferred actions following state $\underline{K}$, and never before; V_0 can therefore be computed (directly or by iterating $\underline{K}$ times equation (12′)) as:

$$V_0 = \sum_{k=0}^{\underline{K}} \delta^k p_k + \delta^{\underline{K}} q'_{\underline{K}}(1-x)(p_{\underline{K}} - \delta p_0) + \delta^{\underline{K}+1}(V - \beta) \equiv f(V). \tag{22}$$

The function f depends on V both directly through the final payoff, and indirectly through speculators' strategy φ'_V, which determines $\underline{K}$ and $q'_{\underline{K}}$ (note that since $\tau \notin \mathbb{N}$, $\underline{K} \neq \tau$, implying by Definition 2.5 that $q'_{\underline{K}}$ is the same for all $\varphi' \in \Omega_B(V)$; the notation $f(V)$ is thus legitimate).

LEMMA 3.1: *For any sequence $(V^n)_{n\in\mathbb{N}}$ converging to $V \in \Gamma$, the sequence of functions $(Q'_{V^n}: t \to Q'_{t,V^n})_{n\in\mathbb{N}}$ converges to $Q'_V: t \to Q'_{t,V}$ on $[\tau - 1, +\infty)$ for the norm of uniform convergence.*

PROOF: Cf. Appendix 3.

Although the integer-valued function $\underline{K}$ is not continuous, this lemma and the recursive properties of Q'_V (cf. Definition 2.3) can be used to prove that f is.

THEOREM 3.1: *The function f is continuous and has a fixed point in Γ. Therefore, there exists a Markov perfect equilibrium of the complete game.*

PROOF: Cf. Appendix 3.

3.2. *Uniqueness*

By Theorem 2.1, there is only one equilibrium corresponding to a given value $V_0 = f(V_0)$.[18] The intuition for why two equilibria with different valuations V_0^1 and V_0^2 cannot exist either is the following. If $V_0^1 > V_0^2$, the firm is more eager to adjust in the first equilibrium than in the second; hence $\overline{T}(V_0^1) < \overline{T}(V_0^2)$. To keep it indifferent during the mixed strategy phase $[\underline{T}, \overline{T}]$, consumers must store more under V_0^1 than under V_0^2; hence $Q'_{V_0^1} \geqslant Q'_{V_0^2}$ everywhere, and $\underline{T}(V_0^1) < \underline{T}(V_0^2)$. Since storage is costly to the firm (in equilibrium), this can be shown to imply, by (22), that the e.p.v. of profits in the first equilibrium is lower than in the second: $f(V_0^1) < f(V_0^2)$, hence a contradiction.

THEOREM 3.2: *The Markov perfect equilibrium of the game is unique,*[21] *up to possible indeterminacies of speculators' strategy at their critical point* τ, *and of the firm's strategy at its thresholds* $\underline{T}$ *and* $\overline{T}$.

PROOF: Cf. Appendix 3.

4. INFLATION, SPECULATION, AND WELFARE

4.1. *Destablizing Speculation*

The effects of speculation on price dynamics are best understood by comparing the equilibrium outcome to the optimal periodic adjustment in the absence of storage (Sheshinski and Weiss (1977)). Let therefore K^{ns} be the state in which the price is adjusted with probability one when no customer can store, i.e. the value of $\underline{K} = K^*$ when $x = 1$ or when $\alpha \geqslant \delta S$.

PROPOSITION 4.1: *In equilibrium*: (i) *if* $K^{ns} \leqslant \text{int}[\tau]$, *then* $\underline{K} = K^* = K^{ns} \leqslant \overline{K}$; (ii) *if* $K^{ns} > \text{int}[\tau]$, *then* $\text{int}[\tau] \leqslant \underline{K} \leqslant K^{ns} \leqslant K^* \leqslant \overline{K}$; (iii) *if* $K^{ns} > \max(\text{int}[\tau], \text{int}[\mu])$, *then* $\underline{K} \leqslant \overline{K} - 1$.

PROOF: Cf. Appendix 4.

These results have a simple interpretation:

PROPERTY 1: If, at the time $K^{ns} + 1$ when the firm would adjust its price in the absence of storage, the magnitude of the price increase does not justify storage by speculators ($K^{ns} + 1 \leqslant \text{int}[\tau] + 1$), this deterministic policy remains optimal (Figure 1 applies, with $\underline{K} = K^{ns}$).

PROPERTY 2: If the price increase at $K^{ns} + 1$ is sufficient to induce storage by speculators but the total loss which they inflict on the firm is not too large ($\text{int}[\tau] + 1 < K^{ns} + 1 \leqslant \text{int}[\mu] + 1$), because for instance there are few of them, adjustment at $K^{ns} + 1$ remains optimal,[22] although all speculators have stored (Figure 3 applies, with $\underline{K} = K^{ns} = \overline{K}$).

PROPERTY 3: If, on the contrary, the threat of speculation is effective ($K^{ns} + 1 > \max(\text{int}[\tau], \text{int}[\mu]) + 1$), the firm must implement a different strategy, leading to either a deterministic adjustment of *shorter period* $\text{int}[\tau] + 1$ (Figure 2

[21] In fact, generally unique: it was assumed that $\tau \notin \mathbb{N}$ and $\mu \notin \mathbb{N}$.
[22] Cf. footnote (17).

with $\underline{K} = \text{int}[\tau]$), or a *randomized* adjustment which also attaches more weight to earlier dates (Figure 4). The price increase will thus generally occur before $K^{ns}+1$, as the outcome of a phase of price uncertainty and increasing amounts of storage by buyers.

In this last case, *speculation* is *destabilizing*, of both prices and quantities, in any sense of the word; Section 4.3 will show that it reduces social welfare as well. Moreover, destabilizing speculation arises here in the absence of any stochastic shocks or informational problem bearing on insurance markets, but from the sole dynamics of imperfect competition.

4.2. *Inflation Causes Price Uncertainty*

A causal relationship between inflation and relative price uncertainty features prominently in macroeconomic discussions of the costs of inflation (cf. Fischer and Modigliani (1978), Fischer (1981a, b), (1984)). This argument is supported by the positive correlation between the two found in many empirical studies (surveyed in Fischer (1981a) and Taylor (1981)), but no theoretical basis has yet been offered for it.[23] The above mechanism, by which an optimal type of noise is injected into price dynamics in order to deter speculation, provides such a foundation.[24]

Since inflation lies here at the origin of both speculative storage and endogenous price uncertainty, the comparative dynamics of the equilibrium with respect to π are of particular interest, in particular to ascertain whether these causal relationships are monotonic. The dependence of the equilibrium on π is unfortunately too complex to allow analytical comparative dynamics exercises, such as can be done with α or β (cf. Bénabou (1986)). Tables I and II present the results of numerical simulations, which point to the following characteristics.

RESULTS OF SIMULATIONS: *As the rate of inflation increases*:

(i) *The support* $[\underline{K}+1,\dots,\overline{K}+1]$ *of the random period* $\tilde{T}$ *of price rigidity separating consecutive price adjustments shifts down by steps; the expectation* $E(\tilde{T})$ *decreases with large enough increases in* π, *but may increase with small ones* (*when the support remains unchanged*).

(ii) *The amount of speculation increases in every period.*

(iii) *The variance of the following period's real price increases with* π *in all periods preceding the occurrence of the adjustment*; *thus*, *more inflation causes more uncertainty*.[25]

[23] In Sheshinski and Weiss (1983) or Caplin and Spulber (1988), deterministic (S, s) rules only amplify the exogenous variability of general inflation.

[24] Even if exogenous uncertainty is generated by parameters such as π or β, the firm must still "process" it so as to optimally deter or reduce storage.

[25] For $\pi \leqslant 5\%$, the equilibrium is of type A (pure strategies, no storage), because $\underline{K} = K^{ns} \leqslant \text{int}[\tau]$; for $\pi = 7\%$, it is of type B.1 (mixed strategies, deterministic outcome without storage), because $\underline{K} = \text{int}[\tau] < K^{ns} = \overline{K}$; in all other cases it is of type B.3 (stochastic outcome). Type B.2 (deterministic outcome, full storage) occurs only for high values of x; for instance, $x = 0.95$ with $\pi = 30\%$ yields $\text{int}[\tau] = 4$, $\underline{K} = K^{ns} = \overline{K} = 11$.

TABLE I[a]

π/k	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	int$[\tau]+1$	$K^{ns}+1$	$E(\tilde{T})$
4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1.00	29	26	26.00
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—			
5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1.00	—	—	—	23	23	23.00
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—			
7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1.00	—	—	—	—	—	—	—	—	—	17	20	17.00
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—			
8	—	—	—	—	—	—	—	—	—	—	—	—	—	—	.96	.04	.004	.001	—	—	—	—	—	—	—	15	19	16.05
	—	—	—	—	—	—	—	—	—	—	—	—	—	.37	.70	.91	1.00	—	—	—	—	—	—	—	—			
10	—	—	—	—	—	—	—	—	—	—	—	—	.90	.09	.01	.004	—	—	—	—	—	—	—	—	—	12	17	14.13
	—	—	—	—	—	—	—	—	—	—	—	.26	.65	.89	1.00	—	—	—	—	—	—	—	—	—	—			
20	—	—	—	—	—	—	—	—	.71	.19	.06	.04	—	—	—	—	—	—	—	—	—	—	—	—	—	7	13	10.44
	—	—	—	—	—	—	—	.36	.77	.98	1.00		—	—	—	—	—	—	—	—	—	—	—	—	—			
30	—	—	—	—	—	—	.65	.20	.08	.01	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5	11	8.56
	—	—	—	—	—	.23	.74	.98	1.00	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—			
50	—	—	—	—	—	.53	.22	.25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	9	7.72
	—	—	—	—	.66	.97	1.00	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—			
100	—	—	—	.52	.21	.27	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	7	5.75
	—	—	.49	.94	1.00	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—			
200	—	—	.50	.50	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	5	4.50
	—	.57	1.00	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—			
300	—	—	.45	.55	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	5	4.55
	—	.83	1.00	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—			
500	—	.53	.47	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	4	3.47
	.50	1.00	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—			
1000	—	.48	.52	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	4	3.52
	.33	1.00	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—			

[a]Variation of the equilibrium outcome with the rate of inflation. For each value of π, the upper line gives the unconditional probability f_k of a price change in each period ($f_{k+1}=(1-q_0)\dots(1-q_{k-1})q_k$ for all $k \geqslant 1$), while the lower lines give the proportion q_k' of speculators who store. The symbol "–" stands for zero. The basic period is a week, but π is given here as an annual rate. The following parameters are fixed: $\beta/S=.25$, $c/S=.0$, $\alpha/S=.02$, $x=.5$, $r=.05$ per year.

For a given inflation rate, the uncertainty faced by buyers also increases over time,[26] until it is suddenly resolved by the occurrence of the price adjustment. Inflation thus generates—and when it increases, exacerbates—*growing price uncertainty*, a *shortening of the price cycle* ($E(\tilde{T}) < K^{ns} + 1$ in general, cf. Table I)), and *mounting speculative storage*. These results give precise meaning to the following description by Buchanan and Wagner (1977) (quoted in Fischer (1984)):

> "Inflation destroys expectations and creates uncertainty; ...it prompts behavioral responses that reflect a general shortening of the time horizon."

4.3. *Some "New" Welfare Costs of Inflation*

The storable nature of most commodities[27] provides each price maker with an incentive to inject noise into the price system (oligopolistic price competition may have similar but weaker effects; cf. Maskin and Tirole (1986)). This incentive could be the source of a price uncertainty externality, which no one likes to experience but all contribute to. Agents were assumed risk neutral, for analytic tractability, but the essential results would remain with risk aversion, since risk averse buyers also store when faced with a certain and large enough price increase. In any case, price uncertainty has costs even in the absence of risk aversion, because it prevents the synchronization of price decisions (e.g. output price adjustment and wage contracting); as a result, relative price distortions and misallocations may propagate themselves across sectors (Blanchard (1983)). The following result also identifies several other costs of inflation, to be added to the lists drawn by Fischer and Modigliani (1978) and Fischer (1981).

PROPOSITION 4.2: *In equilibrium, expected intertemporal social welfare is*:[28]

$$SW_0 = \frac{S-c}{1-\delta} - \frac{\beta + (1-x)(\alpha + c(1-\delta))\Sigma_{k=\underline{K}}^{\overline{K}}\delta^k(1-q_{\underline{K}})\dots(1-q_{k-1})q_k'}{1-\delta\Sigma_{k=\underline{K}}^{\overline{K}}\delta^k(1-q_{\underline{K}})\dots(1-q_{k-1})q_k}. \tag{23}$$

PROOF: Cf. Appendix 4.

The first term is welfare in the absence of inflation; the second one is therefore the total social cost of inflation (and of the speculation it induces) which has

[26] During the mixed strategy phase, the expected price in the next period is, by (13), $(P_t + \alpha)/\delta$, which decreases with both π and t; the two results of increasing uncertainty in Table II therefore also hold if the coefficient of variation, rather than the standard deviation, is used.

[27] Or intertemporal substitution of consumption; cf. footnote 2.

[28] SW_0 is normalized by assuming that the game starts with real price S and zero inventories; other hypotheses are easily dealt with, following the methodology of Section 1.5.

TABLE II[a]

π/k	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
8	—	—	—	—	—	—	—	—	—	—	—	—	—	.48	.74	.94	1.10
10	—	—	—	—	—	—	—	—	—	—	—	.77	1.00	1.19	1.35	—	—
20	—	—	—	—	—	—	—	1.57	1.81	2.03	2.22	—	—	—	—	—	—
30	—	—	—	—	—	1.88	2.19	2.46	2.78	—	—	—	—	—	—	—	—
40	—	—	—	—	2.16	2.52	2.83	—	—	—	—	—	—	—	—	—	—
50	—	—	—	—	2.65	3.02	3.34	—	—	—	—	—	—	—	—	—	—
100	—	—	3.22	3.81	4.30	—	—	—	—	—	—	—	—	—	—	—	—
200	—	4.05	4.91	—	—	—	—	—	—	—	—	—	—	—	—	—	—
300	—	5.03	5.98	—	—	—	—	—	—	—	—	—	—	—	—	—	—
500	4.90	6.31	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
1000	6.45	8.07	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—

[a]Standard deviation of the following week's real price (in %) resulting from various rates of inflation (annual rate, in %). Other parameters have the same values as in Table I.

three components, linked together by the equilibrium strategies (φ, φ'): price adjustment costs, storage costs, and the intertemporal misallocation of production due to speculative purchases. A stochastic equilibrium also gives rise in each period to significant income redistributions between firm and buyers, depending on whether the price increase materializes or not.

5. LONG RUN EQUILIBRIUM, AGGREGATION, AND NONNEUTRALITY

The fact that discontinuous and even randomized individual price policies may arise in response to a constant rate of increase in the general price level raises two important and related questions. The first one was addressed by Caplin (1985) in the context of (S, s) inventory policies, and by Caplin and Spulber (1988) in that of (S, s) real price rules: are individual strategies consistent with the assumed general inflationary process, both individually (a firm's price should increase, on average, at the rate π) and in the aggregate (an index of many such firms' prices should increase at the rate π)? Secondly, how does the cross-sectional distribution of prices among many such sellers depend on inflation, and what are the real effects of this dependence?

5.1. *The Steady State Distribution of Real Prices*

Consider a sector consisting of a continuum (indexed by $\eta \in [0,1]$) of identical monopolistic sellers of nonsubstitutable[29] storable goods, who pursue a common randomized $(S, \tilde{s})$ real price policy with respect to some aggregate price index P^* (for instance the cost of labor).[30] For all $n \in \mathbb{N}$, denote by $h_k(n)$ the proportion of firms in state k, i.e. with a real price of $P_k = S(1+\pi)^{-k}$, in period n; $h(n) \equiv (h_0(n), \dots, h_k(n))$ is thus the cross-sectional distribution of firms' real prices; it could also be interpreted as a prior on the state of a single firm at time n. By the law of large numbers, the dynamics of $h(n)$ are characterized by the same Markov chain which governs the evolution of an individual firm's real price:

$$h(n+1) = h(n) \cdot \underline{M}, \tag{24}$$

$$\underline{M} = \begin{bmatrix} q_0 & 1-q_0 & 0 & \cdot & \cdot & & \cdot & \cdot & 0 \\ q_1 & 0 & 1-q_1 & 0 & \cdot & & \cdot & \cdot & 0 \\ \cdot & & & & & & & \cdot & \cdot \\ q_k & 0 & \cdot & 0 & 1-q_k & 0 & & \cdot & 0 \\ \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ q_{\bar{K}-1} & 0 & \cdot & \cdot & \cdot & & \cdot & \cdot & 1-q_{\bar{K}-1} \\ 1 & 0 & \cdot & \cdot & \cdot & & \cdot & \cdot & 0 \end{bmatrix}. \tag{25}$$

[29] When firms are competing, the adjustment level (here S) becomes an endogenous function of the distribution of prices; cf. Bénabou (1988).

[30] This index may include the firms' own prices, but must also include some outside good(s), if the model is to be consistent in level as well as in growth rate (cf., for instance, equation (27) below).

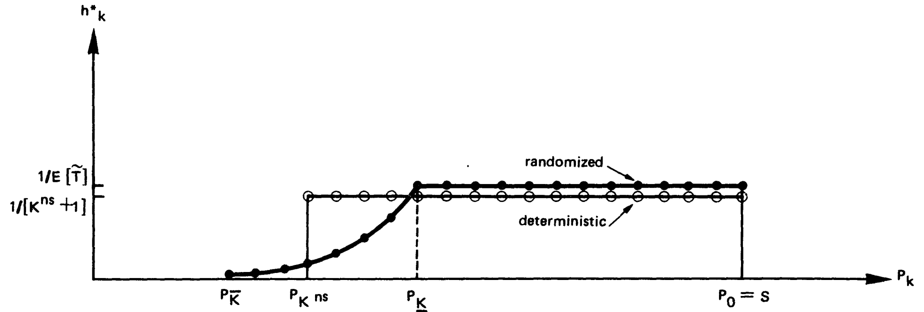

FIGURE 5.—Cross-sectional distribution of real prices in the deterministic (no storage) and randomized cases.

Under certain conditions examined below, the long-run behavior of the system is described by the invariant probability distribution of this Markov chain (cf. Feller (1968)). By (24), a distribution h^* is invariant over time if and only if it is a left-eigenvector of $\underline{M}$ with eigenvalue 1.

PROPOSITION 5.1: *The Markov chain governed by $\underline{M}$ has a unique invariant probability distribution $h^* = (h_0^*, h_1^*, \ldots, h_{\bar{K}}^*)$, defined by*:

$$(26) \qquad (\forall\, k \in \{0, \ldots, \bar{K}\}),\ h_k^* = (1/H) \cdot \prod_{j=0}^{k-1} (1 - q_j),$$

where $q_{-1} \equiv 0$ and

$$H = \sum_{k=0}^{\bar{K}} \prod_{j=0}^{k-1} (1 - q_j) = E[\tilde{T}].$$

PROOF: (26) is straightforward algebra; factoring the terms in the first expression for H yields $H = \sum_{k=0}^{\bar{K}} (k+1) q_k \prod_{j=0}^{k-1} (1 - q_j) = E[\tilde{T}]$. *Q.E.D.*

This steady-state distribution is uniform $(h_k^* = 1/H)$ over the real prices $\{P_0, \ldots, P_{\underline{K}}\}$ belonging to the phase of nonstochastic adjustment (as in Caplin and Spulber (1988)), then decreasing over the real prices $\{P_{\underline{K}+1}, \ldots, P_{\bar{K}}\}$ which are reached through randomized adjustments. Figure 5 illustrates h^* in the no-storage, deterministic case ("ns"), and in the speculation-induced randomized case.[31]

When price strategies are nonstochastic (S, s) rules, the distribution $h(n) = h(0) \cdot \underline{M}^n$ does not generally converge to h^* in the long run, since $\underline{M}$ is then cyclical (of index $\underline{K} + 1$; cf. Varga (1965)). In particular, a positive mass of firms starting with the same real price will remain synchronized forever, generating a

[31] Under deterministic (S, s) rules, the dispersion of real prices is known to increase with the rate of inflation (cf. Bénabou (1988)). Randomization is thus a further source of dispersion (cf. Figure 5); with a more elastic demand, this phenomenon would also occur at the upper limit S (cf. footnote 20).

component of the cross-sectional distribution which cycles through all states and causes any aggregate index to be discontinuous. For randomized $(S, \tilde{s})$ rules, on the contrary, the following important property holds:

THEOREM 5.2: *If firms' common* $(S, \tilde{s})$ *price strategy is randomized, i.e. if* $q_{\underline{K}} < 1$, *the cross-sectional distribution* $h(n)$ *of their real prices converges to the invariant distribution* h^*, *for any initial* $h(0)$.

PROOF: Cf. Appendix 5.

The intuition is simple (more so than the proof): firms sort themselves out through different random drawings of their adjustment dates. Similarly, when $q_{\underline{K}} < 1$, any unconditional prior over the state of an individual firm converges to h^*. With either interpretation, the long-run can thus legitimately be identified with the steady-state distribution.

5.2. *Individual Price Strategies and General Inflation*

Denoting by $E^*[\cdot]$ the expectation operator with respect to the distribution h^*, the average price inflation in a representative firm η's nominal price $P_\eta^N(n)$ will now be computed.

PROPOSITION 5.3: $E^*[\log(P_\eta^N(n+1)/P_\eta^N(n))] = \log(1+\pi)$.

PROOF: Cf. Appendix 5.

As in Caplin and Spulber (1988), who deal with the case of a deterministic (S, s) policy in an exogenous stochastic inflationary environment, $P_\eta^N(n+1)/P_\eta^N(n)$ is therefore a geometric mean-preserving spread of the aggregate inflation rate. Fixed costs models of adjustment thus seem to possess the general property of being *noise-amplifying*, and here, even *endogenously noise-generating*: the resulting individual price dynamics are more noisy than the inflationary process in response to which they arise. This feature contrasts sharply with the noise-dampening characteristics of all models based on misperceptions of nominal and real signals (à la Lucas), on nominal contracts, or on convex adjustment costs, where the only uncertainty in the price system is a fraction of the exogenous noise injected into the economy.

THEOREM 5.4: *If firms' real prices are distributed according to the invariant distribution* h^*, *any index of their nominal prices of the form* $\overline{P}(n) = G[\int_0^1 w(P_\eta^N(n))\,d\eta]$ *which is homogeneous of degree one, grows at the rate* π. *If adjustments are randomized* ($q_{\underline{K}} < 1$), $\overline{P}(n)$ *is asymptotically equivalent to an exponential trend of rate* π, *for any initial distribution of prices.*

PROOF: By homogeneity, and since $h(n)$ is the image of the Lebesgue measure on [0,1] by the random variable $\eta \to P_\eta(n) \in [0, S]$:

$$(27) \qquad \bar{P}(n)/P^*(n) = G\left[\int_0^1 w(P_\eta(n))\, d\eta\right] = G\left[\sum_{k=0}^{\bar{K}} w(P_k) h_k(n)\right].$$

The last term is a constant if $h = h^*$, and in any case converges to that constant when $q_{\underline{K}} < 1$, by Theorem 5.2. *Q.E.D.*

Individual stochastic $(S, \tilde{s})$ pricing strategies thus aggregate back to a price index inflating at the same rate π as the one in response to which they arose. This result (covering in particular the arithmetic and geometric averages) has several important macroeconomic applications.

First, even a constant aggregate rate of inflation, resulting for instance from a constant growth rate of the money supply, can at the same time generate and "cover up" substantial *social costs* at the microeconomic level; these include menu costs, storage costs, distortions in the relative timing of production and sales, and price uncertainty.

Secondly, although price-setters keep pace with inflation in a growth rate sense, inflation *alters the relative prices* between a sector where (S, s) or $(S, \tilde{s})$ rules prevail and the rest of the economy: both P^* and $\bar{P}$ grow at the rate π, but (27) makes clear that the ratio $\bar{P}(n)/P^*(n)$ depends on π. This *nonneutrality* of the inflation rate with respect to relative prices contrasts sharply with Caplin and Spulber's (1988) result that unanticipated money shocks are neutral in an aggregated (S, s) model. It provides a basis for models of the Phillips curve (Naish (1985), Spulber (1987)) or search market equilibrium with inflationary price dynamics (Bénabou (1988)).

Finally, on the transition path to the steady-state distribution (following for instance an unanticipated general inflationary shock), the time varying cross-sectional distribution of real prices will generate an *inventory-driven* (dampened) *cycle* in market demand.

CONCLUSION AND EXTENSIONS

This paper showed in particular that the seller of a storable good in an inflationary environment generally introduces randomness into its price dynamics, while speculators store in increasing numbers, with possibly a final "run" on the good. These results establish that speculation can be destabilizing even with perfect information, and provide a theoretical foundation for the frequent claim that inflation causes price uncertainty.

The mechanism at work here is in fact quite general. Consider a country which tries to peg its exchange rate, but, because of a positive inflation differential with its trading partners, must devaluate repeatedly in order to maintain balanced trade in the long run. Devaluations are costly, politically or because they require international bargaining (as in the European Monetary System), and cannot take place at predictable dates because of speculation. The situation is very similar to the price adjustment problem treated here, and it could be analyzed with a variant of the model, leading most likely to the conclusion that the central bank will pursue an optimally randomized exchange rate policy.

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APPENDIX 1

PROOF OF THEOREM 1: (i, ii) are proved in the text.

(iii) The following result will prove useful many times:

$$\text{(A1)} \qquad \forall\, t \geqslant \tau - 1, \qquad p_{t+1} \leqslant p_\tau = \delta S - \alpha - c < \delta(S - c) = \delta p_0.$$

Let now $t \geqslant 1$ such that $t > \tau, q_{t-1} = 0$ and $q_t = 1$; then $q'_{t-1} = 0$ and $q'_t = 1$ from (ii). Moreover, as $q_{t+1} = 1$ is always feasible, (12′) implies:

$$V_{t+1} \geqslant p_{t+1} + \delta(V - \beta) + q'_{t+1}(1 - x)(p_{t+1} - \delta p_0).$$

Since $q_t = q'_t = 1$, (14) requires:

$$(1 - x)(p_0 - p_{t+1}) = q'_t(1 - x)(p_0 - p_{t+1}) \leqslant V - \beta - V_{t+1} \Rightarrow$$

$$(1 - x)(p_0 - p_{t+1}) \leqslant (1 - \delta)(V - \beta) - p_{t+1} + q'_{t+1}(1 - x)(\delta p_0 - p_{t+1}) \Leftrightarrow$$

$$(1 - \delta)(V - \beta) \geqslant (1 - \delta q'_{t+1})(1 - x)p_0 + (1 - (1 - x)(1 - q'_{t+1}))\,p_{t+1}$$

since $q_t = q'_t = 1$, $V_t = (2 - x)p_t + \delta(V - \beta - p_0(1 - x))$; since $q_{t-1} = q'_{t-1} = 0$, (14) requires:

$$V - \beta - V_t \leqslant 0(p_0 - p_t) \quad \text{or} \quad (1 - \delta)(V - \beta) \leqslant (2 - x)p_t - (1 - x)\delta p_0 \Rightarrow$$

$$(2 - x)p_t - (1 - x)\delta p_0 \geqslant (1 - \delta q'_{t+1})(1 - x)p_0 + (1 - (1 - x)(1 - q'_{t+1}))\,p_{t+1} \Leftrightarrow$$

$$(2 - x)p_t - (1 - x)p_0 \geqslant (1 - q'_{t+1})(1 - x)\delta p_0 + (1 - (1 - x)(1 - q'_{t+1}))\,p_{t+1} > p_{t+1}$$

by (A1). Hence: $F(t; \theta, x) \equiv p_t - (zp_{t+1} + (1 - z)p_0) > 0$, where $z \equiv 1/(2 - x)$. Equivalently, since $p_k = S\theta^{-k} - c$:

$$\mu(\pi, x) \equiv \log[(1 - z/\theta)/(1 - z)]/\log(\theta) > t.$$

Therefore, $q_t = 1$ and $q_{t-1} = 0$ is impossible for $t > \max(\tau, \mu)$. *Q.E.D.*

(iv) The signs of the derivatives of τ are straightforward algebra, and so is that of $\partial\mu/\partial x$. As to $\partial\mu/\partial\pi = \partial\mu/\partial\theta$:

$$\theta(1 - z/\theta)(\log(\theta))^2\, \partial\mu/\partial\theta = \log(\theta)z/\theta + \log[(1 - z)/(1 - z/\theta)](1 - z/\theta)$$
$$< \log\{z\theta/\theta + [(1 - z)(1 - z/\theta)/(1 - z/\theta)]\} = 0$$

because $z/\theta \in (0, 1)$ and the log function is concave. As a consequence:

$$\mu(\theta, x) < \lim_{\theta \to 1^+} (\mu(\theta, x)) = z/(1 - z) = 1/(1 - x). \qquad Q.E.D.$$

NOTATIONS: From here on, let $\sigma(V) \equiv (1 - \delta)(V - \beta)$ and $\Phi(V) \equiv (1 - \delta)(V - \beta - (1 - x)p_0)$; the dependence of σ and Φ on V will be omitted when no confusion results.

PROOF OF LEMMA 2.1: (i) Since $t + 1 < \tau$: $q'_{t+1} = 0$ and: $V_{t+1} \geqslant p_{t+1} + \delta(V - \beta) + 0$. Hence:

$$V - \beta - V_{t+1} \leqslant (1 - \delta)(V - \beta) - p_{t+1} = \sigma - p_{t+1} = p_{T^*+1} - p_{t+1} < 0,$$

so $q_t = 0$ by (14). Q.E.D.

(ii) CLAIM 1: $\forall\ t \in \mathbf{R}_+,\ t \geqslant \overline{T} \Rightarrow q_t > 0$. Indeed, let $q_t = 0$ for such a t.
One cannot have $q_{t+1} > 0$, otherwise:

$$V_{t+1} = p_{t+1} + \delta(V - \beta) + q'_{t+1}(1-x)(p_{t+1} - \delta p_0)$$

and since it is always the case that: $q'_{t+1}(p_{t+1} - \delta p_0) \leqslant 0$ (the first term is zero when $t < \tau$ and the second is negative when $t \geqslant \tau$ by (A1)), this would imply: $V_{t+1} \leqslant p_{t+1} + \delta(V - \beta)$, hence: $V - \beta - V_{t+1} \geqslant \sigma - p_{t+1} > \Phi - p_{\overline{T}+1} = 0$, contradicting (14) with $q_t = 0$.
One cannot have $\forall\ n \in \mathbb{N},\ q_{t+n} = 0$, otherwise:

$$\forall\ n \qquad V_{t+n} = p_{t+n} + \delta V_{t+n+1}$$

so:

$$V_{t+1} = \sum_{k=0}^{+\infty} \delta^k p_{t+1+k} < p_{t+1}/(1-\delta) < V - \beta$$

since $(1-\delta)(V-\beta) = \sigma > \Phi = p_{\overline{T}+1} \geqslant p_{t+1}$. Thus, again: $V - \beta - V_{t+1} > 0$, which contradicts $q_t = 0$. Thus Claim 1 is established.

CLAIM 2: $\forall\ t \in R_+,\ t > \overline{T}$ and $q_t \in (0,1) \Rightarrow q_{t+1} \in (0,1)$. Indeed, if $q_t \in (0,1)$ and $q_{t+1} = 1$ for such a t, then:

$$V_{t+1} = p_{t+1} + \delta(V-\beta) + q'_{t+1}(1-x)(p_{t+1} - \delta p_0), \qquad \text{hence:}$$
$$V - \beta - V_{t+1} = \sigma - p_{t+1} - q'_{t+1}(1-x)(p_{t+1} - \delta p_0) > 0,$$

because the first term is positive since $t + 1 \geqslant \overline{T} + 1 > T^* + 1$, and the second is nonnegative due to (A1). If $t < \tau$, $q'_t = 0$, so (14) and the above inequality imply $q_t = 1$, a contradiction. If $t \geqslant \tau$, $q_{t+1} = 1$ implies $q'_{t+1} = 1$; hence:

$$V - \beta - V_{t+1} = \sigma - (2-x)p_{t+1} + (1-x)\delta p_0$$
$$= q'_t(1-x)(p_0 - p_{t+1}) \leqslant (1-x)(p_0 - p_{t+1})$$

by (14). Hence, $p_{t+1} \geqslant \sigma - (1-\delta)(1-x)p_0 = \Phi = p_{\overline{T}+1}$, contradicting $t > \overline{T}$. Thus Claim 2 is established. To prove the first part of (ii) by contradiction, it is therefore sufficient to show that $q_t \in (0,1)$ is impossible for $t > \max(\overline{T}, \tau)$. Indeed, $q_t \in (0,1)$ would imply: $q_{t+1} \in (0,1)$, $q'_{t+1} = 1$ and:

$$V - \beta - V_{t+1} = \sigma - \delta p_0(1-x) - p_{t+1}(2-x)$$
$$= \Phi - p_{t+1} + (1-x)(p_0 - p_{t+1}) > (1-x)(p_0 - p_{t+1})$$

so $q_t = 1$ by (14), a contradiction. Therefore, for all $t > \overline{T}, q_t = 1$ is the only possible case. We now prove the second part of (ii). For $0 \leqslant t = \overline{T} \leqslant \tau$, $q_{t+1} = 1$ from what precedes, so: $V_{t+1} \leqslant p_{t+1} + \delta(V - \beta)$; hence:

$$V - \beta - V_{t+1} \geqslant \sigma - p_{t+1} > \Phi - p_{t+1} = 0.$$

If $t < \tau, q'_t = 0$ so $q_t = 1$ by (14); if $t = \tau, q_t < 1 = Q_t$ would imply $q'_t = 0$, hence $q_t = 1$, a contradiction. *Q.E.D.*

(iii) By (ii), $q_{t+1} = 1$, and by Claim 1 above, $q_t > 0$. Thus either $q_t = 1$, and then $q'_t = 1$, or $q_t \in (0,1)$, so:

$$V - \beta - V_{t+1} = \sigma - (2-x)p_{t+1} + (1-x)\delta p_0 = q'_t(1-x)(p_0 - p_{t+1}).$$

Therefore $q'_t < 1$ would imply: $\Phi < p_{t+1}$, a contradiction. It is thus necessary that $q'_t = 1$, hence by (13): $q_t \geqslant (P_t + \alpha - \delta P_{t+1})/[\delta(P_0 - P_{t+1})]$. *Q.E.D.*

PROOF OF LEMMA 2.2: (i) Continuity is straightforward. For $t \geqslant \tau, -(\partial Q_t/\partial t)$ has the sign of:

$$\begin{vmatrix} 1/\delta - 1/\theta & \alpha/\delta S \\ -1/\theta & 1 \end{vmatrix} = 1/\delta - (1 - \alpha/\delta S)/\theta > 1/\delta - 1/\theta > 0. \qquad Q.E.D.$$

(ii) The proof of monotonicity and continuity proceeds in three steps. First it is shown by backward induction that:

(A2) $$\forall\ t \in [\tau - 1, \overline{T} + 1), \qquad Q'_t \geqslant (\sigma - p_t)/[(1-\delta)(1-x)p_0],$$

with strict inequality on $[\tau-1,\bar{T})$. Indeed, for $t\in[\bar{T},\bar{T}+1]$: $Q_t'=1$, and the inequality is equivalent to $p_t\geqslant\Phi=p_{\bar{T}+1}$, so the property is true. Suppose now that it holds for $t\geqslant\tau$:

$$\begin{aligned}(1-x)Q_{t-1}' &= [(\delta p_0-p_t)(1-x)Q_t'+\sigma-p_t]/(p_0-p_t)\\ &\geqslant[(\delta p_0-p_t)(\sigma-p_t)/(p_0-\delta p_0)+\sigma-p_t]/(p_0-p_t)\\ &=[(\sigma-p_t)(\delta p_0-p_t+p_0-\delta p_0)]/[(p_0-p_t)(p_0-\delta p_0)]\\ &=(\sigma-p_t)/(p_0-\delta p_0)>(\sigma-p_{t-1})/(p_0-\delta p_0).\end{aligned}$$

Hence the property holds for $t-1$, and (A2) is established. It implies:

(A3) $\quad \forall\, t\in[\tau-1,\bar{T}],\qquad Q_t'>(\sigma-p_0)/[(1-\delta)(1-x)p_0]\equiv\underline{y}.$

Secondly, the function ψ: $(t,y)\to\psi_t(y)$ is not only clearly continuous, but also increasing in both arguments on $[\tau-1,+\infty)\times(\underline{y},+\infty)$. Indeed, it is affine in y, with a positive coefficient by (A1), and decreasing in p_{t+1} because the determinant:

$$\begin{vmatrix}-1-(1-x)y & \sigma+(1-x)y\delta p_0\\ -1 & p_0\end{vmatrix}$$

$$=\sigma-p_0(1+(1-x)(1-\delta)y)=p_0(1-x)(1-\delta)(\underline{y}-y)$$

is negative for $y>\underline{y}$.

Finally, (ii) can now be established by backward induction on successive intervals $I_k\equiv[\max(\tau-1,\bar{T}-k),\bar{T}-(k-1)]$. On the semi-open interval $[\max(\tau-1,\bar{T}-1),\bar{T})$, by definition: $Q_t'=\psi_t(1)$ which is continuous and increasing due to the above properties of ψ (note that $\underline{y}<1$). As the left limit $\psi_{\bar{T}}(1)$ of Q_t' at $\bar{T}$ is easily seen to equal $1=Q_{\bar{T}}'$, these two properties are also true on the closed interval I_1. Suppose now that the proposition holds up to rank k. On $I_{k+1}=[\max(\tau-1,\bar{T}-k-1),\bar{T}-k]$, $Q_t'=\psi_t(Q_{t+1}')$; since Q_{t+1}' is continuous and increasing on I_k, and ψ_t continuous and increasing in both arguments (by (A3) $Q_{t+1}'>\underline{y}$), Q_t' also possesses these two properties. *Q.E.D.*

PROOF OF THEOREM 2.1: We shall make repeated use of the following simple lemma:

LEMMA 2.A: *For any continuation value equilibrium* (φ,φ',V) *and any* $t\in\mathbb{R}_+$: (*i*) $(V-\beta-V_{t+1})/[(1-x)(p_0-p_{t+1})]\leqslant\psi_t(q_{t+1}')$, *with equality when* $q_{t+1}>0$. (*ii*) *If* $q_t\in(0,1)$ *and* $q_{t+1}>0$, *then* $q_t'=\psi_t(q_{t+1}')$. (*iii*) *If* $q_t=1$, *then* $q_t'\leqslant\psi_t(q_{t+1}')$. *If* $q_t=0$ *and* $q_{t+1}>0$, *then* $q_t'\geqslant\psi_t(q_{t+1}')$.

PROOF: Straightforward from (19), (13), and (14).

Since $Q_t=1$ for $t\leqslant\tau$, Lemma 2.1(ii), (iii) and Definition 2.2 imply:

(A4) $\quad\forall\, t\geqslant0,\qquad$ if $t\in(\bar{T},+\infty)$, $q_t=1$; if $t=\bar{T}$, $q_t\in[Q_t,1]$.

(A5) $\quad\forall\, t\geqslant0,\qquad$ if $t\in(\max(\tau,\bar{T}),+\infty)$ or $t=\bar{T}>\tau$, $q_t'=1=Q_t'$.

Thus, there only remains to characterize q_t on $[0,\bar{T})$ and q_t' on $[\tau,\bar{T})$ when $\bar{T}>\tau$, or on $\{\tau\}$ when $\bar{T}\leqslant\tau$ (by definition, $[a,b)\equiv\varnothing$ if $a\geqslant b$).

CASE A: $\bar{T}+1\leqslant\tau$. This implies $T^*<\tau-1$, so $\underline{T}=T^*$ by definition and $q_t=0$ on $[0,T^*)$ by Lemma 2.1(i). Moreover, by (A4), $q_\tau=1$, so by definition of τ, q_τ' can take any value in $[0,1]=[0,Q_\tau']$. It remains to examine q_t for $t\in[T^*,\bar{T}]\cap\mathbb{R}_+$. On $(\max(T^*,\bar{T}-1),\bar{T})\cap\mathbb{R}_+$: $q_{t+1}=1$ and $q_{t+1}'=0$ imply $V_{t+1}=p_{t+1}+\delta(V-\beta)$, hence $V-\beta-V_{t+1}=\sigma-p_{t+1}>0$; thus $q_t=1$ by (14), since $q_t'=0$. Applying this proof to successive intervals $(\max(T^*,\bar{T}-k),\max(T^*,\bar{T}-k+1))\cap\mathbb{R}_+$ yields by induction: $(\forall\, t\in(T^*,\bar{T})\cap\mathbb{R}_+, q_t=1)$. Finally, when $T^*=\underline{T}\geqslant0$: $q_{T^*+1}=1$, $q_{T^*+1}'=0$ so $V-\beta-V_{T^*+1}=\sigma-p_{T^*+1}=\sigma-p_{T^*+1}=0$ and thus $q_{T^*}\in[0,1]=[0,Q_{\underline{T}}]$ is the only restriction imposed on $q_{\underline{T}}=q_{T^*}$ by (13)–(14). This finishes to establish that $(\varphi,\varphi')\in\Omega_0(V)$, with $\underline{T}=T^*<\tau-1$.

CASE B: $\bar{T}+1>\tau$. Note first that: $[0,\bar{T}]=[\tau,\bar{T}]\cup\{(\tau-1,\min(\tau,\bar{T})]\cap\mathbb{R}_+\}\cup[0,\tau-1]$, and that all the intervals on the right-hand side are disjoint. Depending on whether the function Q_t' has a zero on $I\equiv[\tau-1,\bar{T}]$ or not, (cf. Lemma 2.2(ii)), two subcases are possible.

Subcase B1: $\forall\, t \in [\tau - 1, \bar{T})$, $Q'_t \in (0,1)$. By Definition 2.4, $\underline{T}$ is in this case equal to $\min(\tau - 1, T^*)$.

Claim 1:

$$\text{(A6)} \qquad \forall\, t \in (\tau, \bar{T}), \qquad q_t = Q_t, \quad q'_t = Q'_t.$$

The proof is by induction (assuming $(\tau, \bar{T}) \neq \varnothing$). For any t in $I_1 \equiv (\max(\tau, \bar{T} - 1), \bar{T}) \neq \varnothing$, $Q'_{t+1} = 1$ by Definition 2.3; therefore one must have $q_t > 0$, or else: $0 = q'_t \geqslant \psi_t(q'_{t+1}) = \psi_t(1) = Q'_t$ by Lemma 2.A(iii), a contradiction of hypothesis B1. Similarly, $q_t < 1$ or else by the same lemma: $1 = q'_t \leqslant \psi_t(1) = Q'_{t+1}$, another contradiction. Therefore $q_t \in (0,1)$, which by Lemma 2.A(ii) requires $q'_t = \psi_t(q'_{t+1}) = Q'_t \in (0,1)$; this in turn implies $q_t = Q_t$, by (13). Suppose it has been established that:

$$\forall\, t \in I_k \equiv (\max(\tau, \bar{T} - k), \bar{T}), \qquad q_t = Q_t, \qquad q'_t = \psi_t(q'_{t+1}) = Q'_t.$$

Let $t \in I_{k+1}$; since $q_{t+1} = Q_{t+1} > 0$, if q_t were zero, Lemma 2.A(iii) would imply: $0 = q'_t \geqslant \psi_t(q'_{t+1}) = \psi_t(Q'_{t+1}) = Q'_t$, a contradiction; similarly, if $q_t = 1$, then $1 = q'_t \leqslant \psi_t(q'_{t+1}) = \psi_t(Q'_{t+1}) = Q'_t$, again a contradiction. So $q_t \in (0,1)$, which implies $q'_t = Q'_t \in (0,1)$ by Lemma 2.A(ii), and thus $q_t = Q_t$ by (13); this finishes to establishes (A6).

Claim 2:

$$\text{(A7)} \qquad \forall\, t \in (\tau - 1, \min(\tau, \bar{T})] \cap \mathbb{R}_+, \qquad q_t = Q_t = 1; \quad q'_\tau \leqslant Q'_\tau.$$

Let t belong to the above interval. If $t + 1 \geqslant \bar{T}$, $q_{t+1} > 0$ by (A4); if $t + 1 < \bar{T}$, $q_{t+1} > 0$ by (A6). Thus in both cases, by Lemma 2.1(ii):

$$V - \beta - V_{t+1} = (p_0 - p_{t+1})(1 - x)\psi_t(q'_{t+1}) = (p_0 - p_{t+1})(1 - x)Q'_t > 0.$$

Thus $q_t < 1$, implying $q'_t = 0$, would also require $q_t = 1$ by (14), a contradiction. Moreover, from $q_\tau = 1$ and Lemma 2.A(iii): $q'_\tau \leqslant \psi_\tau(q'_{\tau+1}) = Q'_\tau$, i.e. q'_τ can take any value in $[0, Q'_\tau]$. Finally, there only remains to examine q_t for $t \in [0, \tau - 1]$, in the case $\tau \geqslant 1$.

B1.1 If $T^* < \tau - 1$: By Lemma 2.1(i): $\forall t < T^*$, $q_t = 0$. Let us now show the following claim.

Claim 3:

$$\text{(A8)} \qquad \forall\, t \in (T^*, \tau - 1] \cap \mathbb{R}_+, \qquad q_t = 1.$$

Indeed, for $t \in (\max(T^*, \tau - 2), \tau - 1] \cap \mathbb{R}_+$, either $t + 1 > \bar{T}$, so $q_{t+1} = 1$ by (A4), or $t + 1 \in (\tau - 1, \min(\tau, \bar{T})] \cap \mathbb{R}_+$, so $q_{t+1} = 1$ by (A7). Hence:

$$V_{t+1} = p_{t+1} + \delta(V - \beta) + q'_{t+1}(1 - x)(p_{t+1} - \delta p_0) \leqslant p_{t+1} + \delta(V - \beta)$$

by (A1), because the last term is zero unless $t + 1 = \tau$. Therefore: $V - \beta - V_{t+1} \geqslant \sigma - p_{t+1} > 0$. But $q'_t = 0$, so by (14): $q_t = 1 = Q_t$. An induction identical to that of Case A above completes the proof of (A8). Finally, $q_{T^*+1} = 1$ and $q'_{T^*+1} = 0$ imply: $V - \beta - V_{T^*+1} = \sigma - p_{T^*+1} = 0$, so (when $T^* \geqslant 0$) q_{T^*} can take any value in $[0,1] = [0, Q_{T^*}]$. Thus we have shown: $(\varphi, \varphi') \in \Omega_0(V)$, with $\underline{T} = T^* < \tau - 1$.

B1.2 If $T^* = \tau - 1 \geqslant 0$: then $q_t = 0$ on $[0, T^*) = [0, \tau - 1)$ by Lemma 2.1(i). If $\tau > T$, $q_\tau = 1$ by (A4); if $\tau \leqslant \bar{T}$, $q_\tau = 1$ by (A7), hence:

$$\begin{aligned} V - \beta - V_\tau &= (1 - x)(\delta p_0 - p_\tau)\psi_{\tau-1}(q'_\tau) = \sigma - p_\tau + (1 - x)q'_\tau(\delta p_0 - p_\tau) \\ &= (1 - x)q'_\tau(\delta p_0 - p_\tau). \end{aligned}$$

Therefore, either $\psi_{\tau-1}(q'_\tau) = 0$, i.e. $q'_\tau = 0$, and then $(\varphi, \varphi') \in \Omega_0(V)$; or else $\psi_{t-1}(q'_\tau) > 0$, i.e. $q'_\tau > 0$, and then $q_{\underline{T}} = q_{\tau-1} = 1$ by (14), so $(\varphi, \varphi') \in \Omega_1(V)$. In both cases, $\underline{T} = T^* = \tau - 1$.

B1.3 If $T^* > \tau - 1 \geqslant 0$: by Lemma 2.1(i), $q_t = 0$ on $[0, \tau - 1)$. Moreover, by (A7):

$$V - \beta - V_\tau = (1 - x)(\delta p_0 - p_\tau)\psi_{\tau-1}(q'_\tau) = \sigma - p_\tau + (1 - x)q'_\tau(\delta p_0 - p_\tau).$$

Since q'_τ can take any value in $[0, Q'_\tau]$, $V - \beta - V_\tau$ can take any value between $\sigma - p_\tau < 0$ and $(1 - x)(p_0 - p_\tau)Q'_{\tau-1} > 0$ (by hypothesis B1). Therefore, by (14), (φ, φ') belongs to $\Omega_0(V)$ if $\psi_{\tau-1}(q'_\tau) = 0$, to $\Omega_1(V)$ if $\psi_{\tau-1}(q'_\tau) > 0$, and to $\Omega_2(V)$ if $\psi_{\tau-1}(q'_\tau) < 0$, with $\underline{T} = \tau - 1 < T^*$ always.

Subcase B2: $\exists\, !\, \underline{t} \in [\tau - 1, \bar{T})$, $Q'_{\underline{t}} = 0$. This requires: $\tau - 1 \leqslant \underline{t} < T^*$. Otherwise, $m \equiv \max(T^*, \tau - 1)$ satisfies $\delta p_0 - p_{m+1} > 0$, $\sigma - p_{m+1} \geqslant 0$ and $Q'_m \leqslant 0$; but $(1 - x)(p_0 - p_m)Q'_m = \sigma - p_{m+1} + (\delta p_0 - p_{m+1})(1 - x)Q'_{m+1}$, requiring $Q'_{m+1} \leqslant 0$. By induction, this implies: $Q'_{m+2} \leqslant 0, Q'_{m+3} \leqslant 0, \ldots, Q'_{m+k} \leqslant 0$, where $k \equiv \min\{n \in \mathbb{N} | m + n \geqslant \bar{T}\}$. But $Q'_{m+k} = 1$ by definition, hence a contradiction. Since

$T^* \geqslant \tau - 1$, Definition 2.4 then states that $\underline{T} = \underline{t} \in [\tau - 1, T^*)$. The function Q_t' is negative on $[\tau - 1, \underline{T})$ and takes values in $(0,1)$ on $(\underline{T}, \overline{T})$.

(a) Let us first examine $(\underline{T}, \overline{T})$. The induction used in Case B.1 to prove (A6) can be applied to the intervals $J_k \equiv (\max(\tau, \underline{T}, \overline{T} - k), \overline{T})$ to show:

$$\text{(A9)} \qquad \forall\, t \in \left(\max(\tau, \underline{T}), \overline{T}\right), \qquad q_t = Q_t, \qquad q_t' = Q_t'.$$

Similarly, one proves, exactly as for (A7):

$$\text{(A10)} \qquad \forall\, t \in (\underline{T}, \tau] \cap \mathbb{R}_+, \qquad q_t = 1 = Q_t; \qquad q_\tau' \leqslant Q_\tau'.$$

(b) Let us now examine $t = \underline{T}$ (when $\underline{T} \geqslant 0$). If $\underline{T} > \tau - 1$, $q'_{\underline{T}+1} = Q'_{\underline{T}+1}$ by (A4), (A5), and (A9), so: $\psi_{\underline{T}}(q'_{\underline{T}+1}) = Q'_{\underline{T}} = 0$. Hence by Lemma 2.A(i): $V - \beta - V_{\underline{T}+1} \leqslant 0$. Therefore, $q'_{\underline{T}} = 0$, and by (13), $q_{\underline{T}} \in [0, Q_{\underline{T}}]$. If $\underline{T} = \tau - 1$, $q_\tau' \leqslant Q_\tau'$ (by (A10)) implies $\psi_{\tau-1}(q_\tau') \leqslant \psi_{\tau-1}(Q_\tau') = Q'_{\tau-1} = 0$. Therefore, by Lemma 2.A(i):

$$\text{if } q_\tau' = Q_\tau', \text{i.e. } \psi_{\tau-1}(q_\tau') = 0, \quad \text{then} \quad q_{\underline{T}} \in [0, Q_{\underline{T}}];$$

$$\text{if } q_\tau' < Q_\tau', \text{i.e. } \psi_{\tau-1}(q_\tau') < 0, \quad \text{then} \quad q_{\underline{T}} = 0.$$

(c) Finally, it will be established by induction that $q_t = 0$ on $[0, \underline{T})$. For $t \in [\underline{T} - 1, \underline{T}) \cap \mathbb{R}_+$, by (A9) and (A10): $q_{t+1} = Q_{t+1} > 0$ and $q'_{t+1} \leqslant Q'_{t+1}$ (with equality except perhaps at $\tau - 1$). So by Lemma 2.A(i):

$$V - \beta - V_{t+1} = (p_0 - p_{t+1})(1 - x)\psi_t(q'_{t+1}) \leqslant (p_0 - p_{t+1})(1 - x)Q_t' < 0;$$

hence $q_t = 0$ by (14). Assume that the proposition holds on $[\underline{T} - k, \underline{T}) \cap \mathbb{R}_+$, and let $t \in [\underline{T} - k - 1, \underline{T} - k) \cap \mathbb{R}_+$. Then $q_{t+1} = q'_{t+1} = 0$, and by Lemma 2.A(i):

$$V - \beta - V_{t+1} \leqslant (1 - x)(p_0 - p_{t+1})\psi_t(q'_{t+1}) = (1 - x)(p_0 - p_{t+1})\psi_t(0) = \sigma - p_{t+1} < 0$$

since $t + 1 < \underline{T} + 1 < T^* + 1$. Therefore $q_t = 0$, which finishes to prove that, for all $t \in \mathbb{R}_+$: (i) If $t \in [0, \underline{T})$, $q_t = 0$; if $t \in (\underline{T}, \overline{T})$, $q_t = Q_t$; if $t \in (\overline{T}, +\infty)$, $q_t = 1$. (ii) If $t = \underline{T}$, $q_t \in [0, Q_t]$, with the additional restrictions that $q_t = 0$ when $t = \underline{T} = \tau - 1$ and $\psi_{\tau-1}(q_\tau') < 0$, or $q_t = Q_t$ when $t = \underline{T} = \tau - 1$ and $\psi_{\tau-1}(q_\tau') > 0$. (iii) If $t = \overline{T}$, $q_t \in [Q_t, 1]$. (iv) If $t \in [0, \max(\tau, \underline{T}))$, $q_t' = 0$; if $t \in (\max(\tau, \underline{T}), +\infty)$, $q_t' = Q_t'$; if $\underline{T} < \tau$, $q_\tau' \in [0, Q_\tau']$; if $\underline{T} \geqslant \tau$, $q_\tau' = 0$.

Equivalently: (φ, φ') is in $\Omega_1(V)$ if $\underline{T} = \tau - 1 \geqslant 0$ with $\psi_{\tau-1}(q_\tau') > 0$, in $\Omega_2(V)$ if $\underline{T} = \tau - 1 \geqslant 0$ with $\psi_{\tau-1}(q_\tau') < 0$, and in $\Omega_0(V)$ otherwise. The condition $(\varphi, \varphi') \in \Omega(V)$ is therefore necessary for (φ, φ', V) to be a continuation value equilibrium. Since the requirements of (12′), (13), and (14) have been used and exhausted state by state in this proof, this condition is sufficient as well (this is also easy to check directly). Finally, the only cases where $\underline{T} = T^*$ are Cases A, B1.1, and B1.2, where $T^* \leqslant \tau - 1$. *Q.E.D.*

Proof of Proposition 2.2: Let $0 < \max(\tau, \mu) \leqslant \overline{K} \equiv K[\overline{T}]$; then $Q_{\overline{K}} = 1$ and $p_{\overline{K}+1} \leqslant p_{\overline{T}+1} = \Phi$, so:

$$\begin{aligned}(1 - x)(p_0 - p_{\overline{K}})Q'_{\overline{K}-1} &= (\delta p_0 - p_{\overline{K}})(1 - x) + \sigma - p_{\overline{K}} \\ &\geqslant (\delta p_0 - p_{\overline{K}})(1 - x) + p_{\overline{K}+1} + (1 - \delta)(1 - x)p_0 - p_{\overline{K}} \\ &= (1 - x)p_0 + p_{\overline{K}+1} - (2 - x)p_{\overline{K}} = -(2 - x)F(\overline{K}; \theta, x) \geqslant 0\end{aligned}$$

because $\overline{K} \geqslant \mu$ (cf. proof of Theorem 1(iii)). Therefore, $\underline{T} \leqslant \overline{K} - 1$ by Definition 2.4, hence the result. *Q.E.D.*

APPENDIX 3

Proof of Lemma 3.1: Define: $b^n_{t+1} \equiv b_{t+1, V^n}$, $b_{t+1} = b_{t+1, V}$, $\overline{T}^n \equiv \overline{T}(V^n)$, $\psi_t^n \equiv \psi_{t, V^n}$, $\psi_t \equiv \psi_{t, V}$, $Q_t'^n \equiv Q'_{t, V^n}$, and $Q_t' \equiv Q'_{t, V}$. By Definition 2.3, $Q_t'^n$ and Q_t' are equal on [max(τ –

$1, \bar{T}, \bar{T}^n), +\infty)$. Let us compare them on

$$[\tau-1, \max(\bar{T}, \bar{T}^n)) = [\tau-1, \min(\bar{T}, \bar{T}^n)) \cup [\min(\bar{T}, \bar{T}^n), \max(\bar{T}, \bar{T}^n))$$

(when nonempty), for n large enough to have $|\bar{T}^n - \bar{T}| < 1$.

(a) For all $t \in [\tau - 1, \min(\bar{T}^n, \bar{T}))$, by Definition 2.3:

$$Q_t' = \psi_t(Q_{t+1}') = \cdots = \psi_t \circ \cdots \circ \psi_{t+k}(1),$$

where $k \in \mathbb{N}$ is defined by $\bar{T} - 1 \leqslant t + k < \bar{T}$. Similarly:

$$Q_t'^n = \psi_t^n(Q_{t+1}'^n) = \cdots = \psi_t^n \circ \cdots \circ \psi_{t+k^n}^n(1),$$

where $k^n \in \mathbb{N}$ is defined by: $\bar{T}^n - 1 \leqslant t + k^n < \bar{T}^n$. Moreover: $|\bar{T}^n - \bar{T}| < 1$ implies that $|k - k^n| < 2$, so three cases are possible:

$$\text{(1)} \qquad k^n = k: |Q_t'^n - Q_t'| \leqslant a_{t+1}|Q_{t+1}'^n - Q_{t+1}'| + |b_{t+1}^n - b_{t+1}|$$
$$\leqslant a_\infty |Q_{t+1}'^n - Q_{t+1}'| + |b_{t+1}^n - b_{t+1}| \leqslant \cdots$$
$$\leqslant (a_\infty)^{k+1} |Q_{t+1+k}'^n - Q_{t+1+k}'| + \sum_{j=0}^{k} (a_\infty)^j |b_{t+1+j}^n - b_{t+1+j}|$$

by induction, where $a_\infty \equiv (\delta p_0 - p_\infty)/(p_0 - p_\infty) \in (0,1)$ is the limit, reached from below, of the function a_t (cf. Definition 2.2) at $t = +\infty$. But since $k = k^n, Q_{t+1+k}'^n = 1 = Q_{t+1+k}'$ so the first term is zero, while for the second:

$$|b_{t+1+j}^n - b_{t+1+j}| = (1-\delta)|V^n - V|/[(1-x)(p_0 - p_{t+1+j})] \leqslant |V^n - V|/[(1-x)p_0]$$

for all $j \in \mathbb{N}$ and $t \geqslant \tau - 1$, by (A1). Therefore:

$$\text{(A11)} \qquad |Q_t'^n - Q_t'| \leqslant |V^n - V|/[(1-x)(1-a_\infty)p_0].$$

(2) $k^n = k + 1$: The same method applies, but now $Q_{t+1+k}' = 1, Q_{t+1+k}'^n = \psi_{t+1+k}^n(1)$, so:

$$|Q_{t+1+k}'^n - Q_{t+1+k}'| = |(1-\delta)p_0 - (\sigma(V^n) - p_{t+k+2})/(1-x)|/(p_0 - p_{t+k+2})$$
$$= |\Phi(V^n) - p_{t+k+2}|/[(1-x)(p_0 - p_{t+k+2})]$$
$$= |p_{\bar{T}^n+1} - p_{t+k+2}|/[(1-x)(p_0 - p_{t+k+2})].$$

But since $k^n = k+1, \bar{T}^n \leqslant t + k + 2 \leqslant \bar{T}^n + 1$, and $t + k + 2 \geqslant \bar{T} + 1$ so: $0 \leqslant \bar{T}^n + 1 - (t + k + 2) \leqslant \bar{T}^n - \bar{T} < 1$, and:

$$|Q_{t+1+k}'^n - Q_{t+1+k}'| \leqslant |p_{\bar{T}^n+1} - p_{\bar{T}+1}|/[(1-x)(p_0 - p_{t+k+2})]$$
$$= (1-\delta)|V^n - V|/[(1-x)(p_0 - p_{t+k+2})]$$
$$< |V^n - V|/[(1-x)p_0].$$

Adding this term to the right-hand side of (A.11) leads to:

$$\text{(A12)} \qquad |Q_t'^n - Q_t'| \leqslant (1 + 1/(1-a_\infty))|V^n - V|/[(1-x)p_0].$$

(3) $k^n = k - 1$: The induction of case (1) still holds up to rank $k - 1$, so:

$$|Q_t'^n - Q_t'| \leqslant (a_\infty)^k |Q_{t+k}'^n - Q_{t+k}'| + \sum_{j=0}^{k-1} (a_\infty)^j |b_{t+j}^n - b_{t+j}|$$

and

$$|Q_{t+k}'^n - Q_{t+k}'| = |1 - \psi_{t+k}(1)| = |p_{\bar{T}+1} - p_{t+k+1}|/[(1-x)(p_0 - p_{t+k+1})]$$
$$\leqslant |p_{\bar{T}+1} - p_{\bar{T}^n+1}|/[(1-x)(p_0 - p_{t+k+1})] < |V^n - V|/[(1-x)p_0],$$

because $k^n = k - 1$ requires: $\bar{T}^n + 1 \leqslant t + k + 1 \leqslant \bar{T} + 1$; so once again (A12) holds.

(b) For all $t \in [\min(\overline{T}^n, \overline{T}), \max(\overline{T}^n, \overline{T}))$, either:

$$Q_t' = 1, Q_t'^n = \psi_t^n(1), \qquad \text{or} \qquad Q_t'^n = 1, Q_t' = \psi_t(1),$$

according to whether $\overline{T}^n \geqslant \overline{T}$ or $\overline{T} \geqslant \overline{T}^n$; the proofs of (a), Cases 2 and 3 respectively, can then be replicated to finish establishing that (A12) holds for all $t \in [\tau - 1, +\infty)$. *Q.E.D.*

Proof of Theorem 3.1: (a) *Continuity*: Note that it suffices to show separately that $f(V^n)$ converges to $f(V)$ for sequences (V^n) converging to $V \in \Gamma$ from above and from below. The following cases must be distinguished.

Case 1: $\underline{T} < \tau - 1$. It must be that $\underline{T} = T^* < \tau - 1$, so $\underline{K} = K^* \leqslant \text{int}[\tau] < \tau$, $q_{\underline{K}}' = 0$ and:

$$f(V) = \Sigma_{j=0}^{\underline{K}} \delta^j p_j + \delta^{\underline{K}+1}(V - \beta).$$

(1.1) $p_{K^*} > \sigma(V) > p_{K^*+1}$; for n large enough, $p_{K^*} > \sigma(V^n) > p_{K^*+1}$ so K^{*n} is equal to K^*; hence $\underline{K}^n = \underline{K}$, so $f(V^n) - f(V) = \delta^{\underline{K}+1}(V^n - V)$ which yields the result.

(1.2) $p_{K^*} > \sigma(V) = p_{K^*+1}$; equivalently, $K^* = T^* < \tau - 1$. If (V^n) converges to V from above, $K^{*n} = K^*$, and one is back in Case 1.1. From here on in the rest of Case 1 it will therefore be assumed that $V^n < V$ for all n, which implies that $K^{*n} = K^* + 1$ for n large enough.

(1.2.1) $K^* + 1 \leqslant \text{int}[\tau]$; $f(V^n) = \Sigma_{j=0}^{\underline{K}+1} \delta^j p_j + \delta^{\underline{K}+2}(V^n - \beta)$ converges, as $n \to +\infty$, to:

$$\sum_{j=0}^{\underline{K}} \delta^j p_j + \delta^{\underline{K}+1}[p_{K^*+1} + \delta(V - \beta)]$$

$$= \sum_{j=0}^{\underline{K}} \delta^j p_j + \delta^{\underline{K}+1}[(1-\delta)(V-\beta) + \delta(V - \beta)] = f(V).$$

(1.2.2) $K^* = \text{int}[\tau]$, so $K^{*n} = \text{int}[\tau] + 1$; since $\underline{T} < \tau - 1$, it must be (cf. proof of Theorem 2.1, subcase B2) that: $(\forall t \in [\tau - 1, \overline{T}]$: $Q_t' > 0)$. Lemma 3.1 then implies that, for large enough n: $(\forall t \in [\tau - 1, \overline{T}], Q_t'^n > 0)$. Therefore, $\underline{T}^n \leqslant \tau - 1$, while on the other hand: $V^n < V$, so $T^{*n} > T^* = K^* = \text{int}[\tau] > \tau - 1$. Definition 2.4 then requires $\underline{T}^n = \tau - 1$, $\underline{K}^n = \text{int}[\tau] = K^* = \underline{K}$, and convergence is again immediate.

Case 2: $\tau - 1 \leqslant \underline{T} < \text{int}[\tau]$. This implies that $\underline{K} = \text{int}[\tau] > \tau - 1$, and therefore: $\forall\, t \in [\text{int}[\tau], \overline{T}], Q_t' > 0$. Then by Lemma 3.1, for n large enough: $\forall t \in [\text{int}[\tau], \overline{T}], Q_t'^n > 0$, therefore $\underline{K}^n \leqslant \text{int}[\tau]$. On the other hand, $\underline{T} \geqslant \tau - 1 > \text{int}[\tau] - 1$, so for n large enough, $\underline{T}^n > \text{int}[\tau] - 1$, $\underline{K}^n > \text{int}[\tau] - 1$; hence $\underline{K}^n = \text{int}[\tau] = \underline{K}$, and convergence is again immediate.

Case 3: $\underline{T} > \text{int}[\tau]$. This requires: $\underline{K} \geqslant 1$ and $\exists\,! \,\underline{T}, \text{int}[\tau] < \underline{T} < T, Q_{\underline{T}}' = 0$.

(3.1) If $\underline{T} \notin \mathbb{N}$, or equivalently, $Q_{\underline{K}-1}' < 0, Q_{\underline{K}}' > 0$. By Lemma 3.1, for n large enough, $Q_{\underline{K}-1}'^n < 0$ and $Q_{\underline{K}}'^n > 0$, hence $\underline{K}^n = \underline{K}$ and:

$$f(V^n) = \sum_{j=0}^{\underline{K}} \delta^j p_j + \delta^{\underline{K}}(p_{\underline{K}} - \delta p_0)(1 - x) Q_{\underline{K}}'^n + \delta^{\underline{K}+1}(V^n - \beta)$$

which converges to:

$$\sum_{j=0}^{\underline{K}} \delta^j p_j + \delta^{\underline{K}}(p_{\underline{K}} - \delta p_0)(1 - x) Q_{\underline{K}}' + \delta^{\underline{K}+1}(V - \beta) = f(V).$$

(3.2) If $\underline{T} \in \mathbb{N}$, equivalently, $Q_{\underline{K}}' = 0$; if (V^n) converges from above, it is easy to verify (by induction) that $\forall t \in [\tau - 1, \overline{T}), Q_t'^n > Q_t'$, in particular: $Q_{\underline{K}}'^n > Q_{\underline{K}}' = 0$. Moreover, Lemma 3.1 still implies, for n large enough: $Q_{\underline{K}-1}'^n < 0$; therefore $\underline{K}^n = \underline{K}$, which brings us back to Case 3.1. From here on, it will be assumed that $(\forall n, V^n < V)$, implying that $Q_{\underline{K}}'^n < 0$. But since $\underline{K} + 1 > \tau$: $Q_{\underline{K}+1}' > Q_{\underline{K}}' = 0$, hence by Lemma 3.1, for n large enough: $Q_{\underline{K}+1}'^n > 0$. Thus $\underline{K}^n = \underline{K} + 1$, and:

$$f(V^n) = \sum_{j=0}^{\underline{K}} \delta^j p_j + \delta^{\underline{K}+1} Q_{\underline{K}+1}'^n (1 - x)(p_{\underline{K}+1} - \delta p_0) + \delta^{\underline{K}+2}(V^n - \beta)$$

which converges to:

$$\sum_{j=0}^{\underline{K}} \delta^j p_j + \delta^{\underline{K}+1}[p_{\underline{K}+1} + Q'_{\underline{K}+1}(1-x)(p_{\underline{K}+1} - \delta p_0) + \delta(V-\beta)]$$

$$= \sum_{j=0}^{\underline{K}} \delta^j p_j + \delta^{\underline{K}+1}[V - \beta - (1-x)(p_0 - p_{\underline{K}+1})\psi_{\underline{K}}(Q'_{\underline{K}+1})].$$

Since $\psi_{\underline{K}}(Q'_{\underline{K}+1}) = Q'_{\underline{K}} = 0$, this last expression is equal to $f(V)$.

CASE 4: $\underline{T} = \text{int}[\tau]$. Equivalently: $Q'_{\text{int}[\tau]} = 0$. If V^n converges to V from above, $Q'^n_{\text{int}[\tau]} > 0$, so $\underline{K}^n \leqslant \text{int}[\tau]$; moreover, $\underline{T} = \text{int}[\tau] > \tau - 1$, which requires: $T^* > \tau - 1$, hence $T^{*n} > \tau - 1, \underline{T}^n \geqslant \tau - 1$, $\underline{K}^n \geqslant \text{int}[\tau]$. Thus $\underline{K}^n = \text{int}[\tau] = \underline{K}$, and the result is immediate. Now if V^n converges from below, then for n large enough, $\underline{K}^n = \text{int}[\tau] + 1 = \underline{K} + 1$, and the proof is the same as in Case 3.2. *Q.E.D.*

(b) *Fixed point*: By (22), for all $V \in \Gamma = [(p_0 - \delta\beta)/(1-\delta), p_0/(1-\delta)]$:

$$f(V) \leqslant [(1-\delta^{\underline{K}+1})/(1-\delta)]p_0 + \delta^{\underline{K}+1}V \leqslant p_0/(1-\delta).$$

By construction, $f(V)$ is the payoff obtained by the firm in the continuation value game under its optimal strategy $\varphi \in \mathscr{F}$ (given that customers play $\varphi' \in \mathscr{F}$). It is therefore at least equal to the payoff received by adjusting in state 0, given $\varphi' \in \mathscr{F}$ (note that $q'_0 = 0$):

$$f(V) \geqslant p_0 + \delta(V-\beta) \geqslant p_0 + \delta(p_0 - \beta)/(1-\delta) = (p_0 - \delta\beta)/(1-\delta).$$

Thus f is continuous and maps Γ into itself, hence the result. *Q.E.D.*

PROOF OF THEOREM 3.2: Let there be two equilibria with initial firm valuations V_0^1 and V_0^2—in short V^1, V^2—with $V^1 > V^2$. Then $\sigma(V^2) < \sigma(V^1)$, $\Phi(V^2) < \Phi(V^1)$, hence $\overline{T}(V^2) > \overline{T}(V^1)$, and by a straightforward induction: $\forall t \in [\tau - 1, +\infty)$: $Q'_{t,V^1} \geqslant Q'_{t,V^2}$, from which follows: $(\forall t \in R_+$: $q'_{t,V^1} \geqslant q'_{t,V^2})$ where q'_{t,V^j}, $j \in \{1,2\}$, denotes speculators' strategy in the equilibrium with firm valuation V^j. This implies in turn: $\underline{T}^1 \equiv \underline{T}(V^1) \leqslant \underline{T}(V_2) \equiv \underline{T}^2$, and thus $\underline{K}^1 \leqslant \underline{K}^2$. In particular:

$$\forall\, k < \underline{K}^1, \quad 0 = q'_{k,V^1} = q'_{k,V^2} \quad \text{and} \quad q'_{\underline{K}^1,V^1} \geqslant q'_{\underline{K}^1,V^2} \geqslant 0.$$

Hence: $q'_{\underline{K}^1,V^1}(\delta p_0 - p_{\underline{K}^1}) \geqslant q'_{\underline{K}^1,V^2}(\delta p_0 - p_{\underline{K}^1})$, because $\delta p_0 - p_{\underline{K}^1} \geqslant 0$ unless $\underline{K}^1 < \tau$, but then $q'_{\underline{K}^1,V^1} = 0$. Therefore, (22) and $V^1 = f(V^1)$ imply:

$$V^1 - \beta = \left[\sum_{k=0}^{\underline{K}^1} \delta^k p_k + \delta^{\underline{K}^1} q'_{\underline{K}^1,V^1}(1-x)(p_{\underline{K}^1} - \delta p_0) - \beta\right] \Big/ \left(1 - \delta^{\underline{K}^1+1}\right)$$

$$\leqslant \left[\sum_{k=0}^{\underline{K}^1} \delta^k p_k + \delta^{\underline{K}^1} q'_{\underline{K}^1,V^2}(1-x)(p_{\underline{K}^1} - \delta p_0) - \beta\right] \Big/ \left(1 - \delta^{\underline{K}^1+1}\right).$$

The last term is the firm's payoff (minus β) under a strategy of periodic adjustments in state $\underline{K}^1$ (with probability one), given speculators' strategy $\varphi'_{V^2} \in \mathscr{F}$. By definition, it is no greater than its payoff from the optimal strategy given $\varphi'_{V^2} \in \mathscr{F}$, i.e., the equilibrium payoff (minus β) $V^2 - \beta$. Hence $V^1 - \beta \leqslant V^2 - \beta$, a contradiction. *Q.E.D.*

APPENDIX 4

PROOF OF PROPOSITION 4.1: Some preliminary results on the firm's intertemporal payoff under various strategies—in the presence and in the absence of speculation—must first be established. For all $k \in N$, let $M(k)$ denote this payoff when speculators play their equilibrium strategies $\varphi' \in \mathscr{F}$ but the firm adjusts its price periodically (with probability one) when state k is reached:

$$\text{(A13)} \qquad M(k) \equiv \left[\sum_{j=0}^{k-1} \delta^j \big(p_j + q_j'(1-x)(p_j - \delta p_{j+1}) \big) + \delta^k \big(p_k + q_k'(1-x)(p_k - \delta p_0) \big) - \beta \right] \Big/ (1 - \delta^{k+1}).$$

Since $q_j' = 0$ for $j < \underline{K}$ and $V_0 = f(V_0)$, (22) can be rewritten: $V_0 - \beta = M(\underline{K})$. Thus periodic adjustment at $\underline{K}$ is optimal (given $q' \in \mathscr{F}$). Moreover:

$$\text{(A14)} \qquad \forall\, k \leqslant \underline{K}, \quad M(k) \leqslant M(\underline{K}), \text{ with strict inequality for } k < \underline{K},$$

because adjustment in a state $k < \underline{K}$ is strictly suboptimal (given $q' \in \mathscr{F}$) since $V_0 - \beta - V_{k+1} \leqslant (1-x)(p_0 - p_{k+1})Q_k' < 0$ by Lemma 2.A(i) and Definition 2.5. Consider now the limiting no-storage case ($x = 1$), with all variables superscripted by "ns". The system (13)–(14) is then identical to what it would be with $x \in (0,1)$ but $\alpha \geqslant \delta S$ (i.e. $\tau = +\infty$):

$$\text{(13')} \qquad \forall\, t \geqslant 0, \quad q_t' = 0.$$

$$\text{(14')} \qquad \forall\, t \geqslant 0, \quad q_t = 0, \quad q_t = 1, \quad \text{or} \quad q_t \in [0,1], \quad \text{according to whether}$$
$$V_0 - \beta - V_{t+1} < 0, \quad V_0 - \beta - V_{t+1} > 0 \quad \text{or} \quad V_0 - \beta - V_{t+1} = 0.$$

Indeed, $x = 1$ or $\alpha \geqslant \delta S$ both mean that no one can ever store profitably. Therefore, when $x = 1$, the game still has a unique solution, which is the same as when $\alpha \geqslant \delta S$, and yields an equilibrium payoff V_0^{ns} to the firm. It is easily seen, from Definitions 2.1 and 2.4, that this equilibrium is also the limit of the solutions with $\alpha < \delta S$ and x tending to 1 from below, and that: $\underline{T}^{ns} = \overline{T}^{ns} \equiv T^{ns}$, $\underline{K}^{ns} = \overline{K}^{ns} \equiv K^{ns}$. For this equilibrium, (A14) yields:

$$\text{(A15)} \qquad \forall\, k \leqslant K^{ns}, \qquad M^{ns}(k) \leqslant M^{ns}(K^{ns}),$$

with strict inequality for $k < K^{ns}$, where $M^{ns}(k)$ is the firm's payoff to periodic adjustment in state k, given that customers never store. Note that $M^{ns}(k)$ is given by (A13) with all q_j''s replaced by zero; in particular, $M(\underline{K}) \leqslant M^{ns}(\underline{K})$, with equality if and only if $q_{\underline{K}}' = 0$. It will now be shown that:

$$\text{(A16)} \qquad \underline{K} \leqslant K^{ns} \leqslant K^*.$$

Indeed, if $K^{ns} < \underline{K}$: $M^{ns}(K^{ns}) = M(K^{ns}) < M(\underline{K}) \leqslant M^{ns}(\underline{K})$, contradicting (A15). Moreover:

$$p_{T^{ns}} + 1 = (1-\delta)(V_0^{ns} - \beta) = (1-\delta) M^{ns}(K^{ns})$$
$$\geqslant (1-\delta) M^{ns}(\underline{K}) \geqslant (1-\delta) M(\underline{K}) = \sigma = p_{T^*+1},$$

implying $T^{ns} \leqslant T^*$, and $K^{ns} \leqslant K^*$. Proposition 4.1 can now be proven.

(i) When $K^{ns} \leqslant \text{int}[\tau]$, then $\underline{K} \leqslant K^{ns} \leqslant \text{int}[\tau]$ by (A16) and two cases arise: (a) $\underline{K} \leqslant \text{int}[\tau] - 1$, implying $\underline{T} < \tau - 1$, which by definition requires $\underline{T} = T^* < \tau - 1$. Hence $\underline{K} = K^* = K^{ns}$, by (A16). (b) $\underline{K} = \text{int}[\tau] < \tau$; since $K^{ns} \leqslant \text{int}[\tau]$, (A16) then requires $\underline{K} = K^{ns}$, hence:

$$p_{T^{ns}+1} = (1-\delta) M^{ns}(K^{ns}) = (1-\delta) M^{ns}(\underline{K}) = (1-\delta) M(\underline{K}) = \sigma = p_{T^*+1},$$

because $\underline{K} < \tau$ implies $q_{\underline{K}}' = 0$. Thus: $T^{ns} = T^*$, $K^* = K^{ns} = \underline{K}$. *Q.E.D.*

(ii) When $K^{ns} > \text{int}[\tau]$, assume that $\underline{K} < \text{int}[\tau]$; as in (a) above, this implies $\underline{K} = K^*$; but (A16) then requires $\underline{K} \geqslant K^{ns} > \text{int}[\tau]$, a contradiction. *Q.E.D.*

(iii) Results directly from Proposition 2.2, since $\overline{K} \geqslant K^* \geqslant K^{ns}$. *Q.E.D.*

PROOF OF PROPOSITION 4.2: Define for all $k \in \mathbb{N}$: $Y_k \equiv p_k + q_k'(1-x)[p_k - \delta(q_k p_0 + (1-q_k)p_{k+1})]$. Since the price is adjusted with probability q_k in state k, and $q_{\overline{K}} = 1$, (12) yields:

$$\text{(A17)} \qquad V_0 - \beta = \frac{\Sigma_{k=0}^{\overline{K}} \delta^k (1-q_0) \cdots (1-q_{k-1}) Y_k - \beta}{1 - \delta \Sigma_{k=0}^{\overline{K}} \delta^k (1-q_0) \cdots (1-q_{k-1}) q_k},$$

where $q_{-1} \equiv 0$ by convention. Similarly, defining W_0 (resp. W_0') as the average expected present value of speculators' (resp. nonspeculators') utility in state zero (with no initial stocks) and, for all $k \in \mathbb{N}$:

$Z_k \equiv S - P_k + q_k'\{-\alpha - P_k + \delta[q_k P_0 + (1-q_k)P_{k+1}]\}$, one can compute:

$$W_0 = \frac{\sum_{k=0}^{\bar{K}} \delta^k (1-q_0)\cdots(1-q_{k-1})Z_k}{1-\delta\sum_{k=0}^{\bar{K}} \delta^k (1-q_0)\cdots(1-q_{k-1})q_k}, \tag{A18}$$

$$W_0' = \frac{\sum_{k=0}^{\bar{K}} \delta^k (1-q_0)\cdots(1-q_{k-1})(S-P_k)}{1-\delta\sum_{k=0}^{\bar{K}} \delta^k (1-q_0)\cdots(1-q_{k-1})q_k}. \tag{A19}$$

Total intertemporal social welfare (net of the first adjustment cost) is $SW_0 = xW_0' + (1-x)W_0 + V_0 - \beta$, or:

$$SW_0 = \frac{\sum_{k=0}^{\bar{K}} \delta^k (1-q_0)\cdots(1-q_{k-1})[x(S-P_k)+(1-x)Z_k + Y_k] - \beta}{1-\delta\sum_{k=0}^{\bar{K}} \delta^k (1-q_0)\cdots(1-q_{k-1})q_k}.$$

But, for all $k \in \mathbb{N}$: $x(S-P_k)+(1-x)Z_k+Y_k = S - c - (1-x)q_k'(\alpha + c(1-\delta))$, and:

$$\begin{aligned}(1-\delta)\sum_{k=0}^{\bar{K}} \delta^k(1-q_0)\cdots(1-q_{k-1}) &= 1 + \sum_{k=1}^{\bar{K}} \delta^k (1-q_0)\cdots(1-q_{k-1}) \\ &\quad - \sum_{k=0}^{\bar{K}} \delta^{k+1}(1-q_0)\cdots(1-q_{k-1})(1-q_k+q_k) \\ &= 1 - \delta\sum_{k=0}^{\bar{K}} \delta^k(1-q_0)\cdots(1-q_{k-1})q_k, \qquad \text{so that:}\end{aligned}$$

$$SW_0 = \frac{S-c}{1-\delta} - \frac{\beta + (1-x)(\alpha + c(1-\delta))\sum_{k=0}^{\bar{K}} \delta^k(1-q_0)\cdots(1-q_{k-1})q_k'}{1-\delta\sum_{k=0}^{\bar{K}} \delta^k(1-q_0)\cdots(1-q_{k-1})q_k}, \tag{A20}$$

which, given that $q_k = 0$ for $k < \underline{K}$, completes the proof. *Q.E.D.*

APPENDIX 5

PROOF OF THEOREM 5.2:

CLAIM 1: All eigenvalues of $\underline{M}$ have modulus no greater than 1. Indeed, for any complex matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant N}$ (cf. Varga (1965, p. 17)):

$$|v| \leqslant \max\left\{\sum_{i=1}^{N} |a_{ij}|, 1 \leqslant j \leqslant N\right\}$$

where v is any eigenvalue of A. Since the second term is equal to 1 for $A = \underline{M}$ (with $N = \bar{K}+1$), and $h^* \cdot \underline{M} = h^*$, $\underline{M}$ has radius of convergence equal to one.

CLAIM 2: $\underline{M}$ is noncyclic, or primitive (i.e. it has only one eigenvalue equal to its radius of convergence) if and only if $q_{\underline{K}} < 1$.

The proof rests on the following theorem (cf. Varga (1965, p. 44)):

THEOREM (FROBENIUS): *An irreducible square matrix A with nonnegative coefficients is cyclic of index $n(A)$ (i.e. has $n(A)$ eigenvalues with modulus equal to its radius of convergence), where $n(A)$ is the greatest common denominator of the differences in successive degrees appearing with nonzero coefficients in its characteristic polynomial.*

The matrix $\underline{M}$ has nonnegative elements. By definition (Varga (1965, p. 20)) it is irreducible if and only if it has a strongly connected graph, i.e., for any pair $(i,j) \in \{1,\ldots,\bar{K}+1\}^2$ there exists a sequence $(i_0 = i, i_1), (i_1, i_2), \ldots, (i_n, i_{n+1} = j)$ such that the corresponding coefficients of $\underline{M}$ are nonzero (this will be denoted as $i \sim j$). By definition of $\bar{K}$, for any $i < \bar{K}$, $m_{i,i+1} = 1 - q_{i-1} > 0$, therefore $i \sim j$ for any (i, j) with $j > i$. Moreover, $m_{\bar{K}+1,1} = 1$, so $i \sim 1$ for all i; but $1 \sim j$ for all $j > 1$, hence the result. Let us now compute from (25) the characteristic polynomial of $\underline{M}$:

$$G(X) \equiv -X^{\bar{K}+1} + X^{\bar{K}} q_0 + X^{\bar{K}-1}(1-q_0)q_1 + \cdots$$
$$+ \cdots + X(1-q_0)\cdots(1-q_{\bar{K}-2})q_{\bar{K}-1} + (1-q_0)\cdots(1-q_{\bar{K}-1}).$$

Moreover, since $q_k = 0$ when $k < \underline{K}$, $q_k \in (0,1)$ if $\underline{K} \leqslant k < \bar{K}$, and $q_{\bar{K}} = 1$:

$$G(X) = -X^{\bar{K}+1} + q_{\underline{K}} X^{\bar{K}-\underline{K}}$$
$$+ (1-q_{\underline{K}})\big[X^{\bar{K}-\underline{K}-1} q_{\underline{K}+1} + X^{\bar{K}-\underline{K}-2}(1-q_{\underline{K}+1})q_{\underline{K}+2} + \cdots$$
$$+ \cdots + (1-q_{\underline{K}+1})(1-q_{\underline{K}+2})\cdots(1-q_{\bar{K}-1})\big].$$

If $0 < q_{\underline{K}} < 1$, $G(X)$ has nonzero coefficients on $\bar{K} - \underline{K}$ and $\bar{K} - \underline{K} - 1$, so $n(\underline{M}) = 1$ by Frobenius' Theorem; if $q_k = 1$ $(\underline{K} = \bar{K})$, then $n(\underline{M}) = \bar{K} + 1$, hence Claim (2).

To conclude the proof of the theorem, let $q_{\underline{K}} < 1$, and consider the hyperplane

$$L \equiv \left\{ z \in \mathbb{R}^{\bar{K}+1} \mid \Sigma_{k=0}^{\bar{K}+1} z_k = 0 \right\}.$$

For any probability distribution h, $h - h^* \in L$. Moreover, $[h \to h \cdot \underline{M}]$ maps L into itself, therefore:

$$\forall h \in \mathbb{R}^{\bar{K}+1}, \quad \exists u \in L, \quad \forall n \in N, \quad |h \cdot \underline{M}^n - h^*| \leqslant B^n |u|$$

where B is the radius of convergence of $\underline{M}$'s restriction to L. But, since $h^* \notin L$, Claims 1 and 2 imply: $|B| < 1$, hence the result. *Q.E.D.*

PROOF OF PROPOSITION 5.3: With probability h_k^*, the firm is in state k; $\log(P_\eta^N)$ then increases by zero with probability $1 - q_k$ and by $\log[(1+\pi)^{k+1}]$ with probability q_k; thus:

$$E^*\left[\log\left(P_\eta^N(t+1)/P_\eta^N(t)\right)\right]/\log(1+\pi)$$
$$= \sum_{k=0}^{\bar{K}} h_k^* q_k (k+1)$$
$$= \left[\sum_{k=\underline{K}}^{\bar{K}} (1-q_0)\cdots(1-q_{k-1}) q_k (k+1)\right] \Big/ H = E[\tilde{T}]/H = 1. \qquad Q.E.D.$$

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