Technical Appendix for "Willpower and Personal Rules"

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Proof of Proposition 1. Consider first the weak type's probability of perseverance at date 1.

Pooling: $q_1 = 1$. Then $\rho_2^+ = \rho_1$, while ρ_2^- can be any $\rho' \leq \rho$. Optimality in (3) then requires $\rho_1 \geq \rho_2^* > \rho'$, otherwise the right-hand side would be zero. Let therefore $\rho_1 > \rho_2^*$ (leaving aside the measure-zero case where $\rho_1 = \rho_2^*$). Given that $c/\beta_L < C(\lambda)$, this is indeed an equilibrium.

Semi-separation: $q_1 \in (0,1)$. This implies $\rho_2^+ \in (\rho_1, 1)$ and $\rho_2^- = 0$. Furthermore, (3) must now hold with equality, $c/\beta_L = B - b + \delta\lambda \left[V_2^L(\rho_2^+) - a\right]$. This can only occur if

$$\rho_2^+ \equiv \frac{\rho_1}{\rho_1 + (1-\rho_1)(q_1 + (1-q_1)(1-\lambda))} = \rho_2^*, \tag{A.1}$$

requiring $\tilde{\rho}_1(\lambda) < \rho_1 < \rho_2^*$, and if the mixing probability $p_2^* \equiv p_2(\rho_2^*)$ that will result in period 2 satisfies $c/\beta_L = B - b + \delta\lambda p_2^*(b-a)$. This condition and the one above uniquely determine q_1 and p_2^* in $[0,1]$ as given in Proposition 1.

Separation: $q_1 = 0$. This implies again that $\rho_2^- = 0$, and thus one must have $c/\beta_L \geq B - c + \delta\left[V_2^L(\rho_2^+) - a\right] = V_2^L(\rho_2^+) - a$. With $c/\beta_L < C(\lambda)$ this can only happen for $\rho_2^+ < \rho_2^*$, which means that $\rho_1 < \tilde{\rho}_1(\lambda)$.

Finally, we turn to the individual's task selection in period 1. For $\rho_1 \geq \rho_2^*$ both types choose P with probability 1, so it is optimal to select W. Indeed, this yields $B - c$ in period 1 and $\delta\left[\rho_1(B-c) + (1-\rho_1)b\right]$ in period 2, against a/γ in period 1 and the same expected payoff in period 2 if NW is chosen instead (there is then no new information, so $\rho_2 = \rho_1$ and W is chosen in period 2). Consider now the case where $\tilde{\rho}_1(\lambda) < \rho_1 < \rho_2^*$. Choosing W rather than NW then leads to expected net gains of Δ_1 in period 1 and Δ_2 in period 2, where:

$$\Delta_1 \equiv \rho_1\left(B - c - a/\gamma\right) + (1-\rho_1)\left[q_1\left(B-c\right) + (1-q_1)\, b - a/\gamma\right] \tag{A.2}$$

is increasing in ρ_1, both directly and through q_1, and the same is true for

$$\begin{aligned}
\Delta_2/\delta \quad &\equiv \quad \rho_1\left[p_2^*(B-c) + (1-p_2^*)a)\right] + \\
&\qquad (1-\rho_1)\left\{\left[q_1 + (1-q_1)(1-\lambda))\right]\left[p_2^* b + (1-p_2^*)a)\right] + (1-q_1)\lambda a\right\} - a \\
&= \quad p_2^*\left\{\rho_1(B-c-a) + (1-\rho_1)\left[q_1 + (1-q_1)(1-\lambda)\right](b-a)\right\}.
\end{aligned} \tag{A.3}$$

By continuity, the total gain $\Delta_1 + \Delta_2$ positive just below $\rho_1 = \rho_2^*$. Therefore, the choice between W and NW in period 1 is indeed governed by a cutoff $\rho_1^* < \rho_2^*$. It is ambiguous, on the other hand, whether ρ_1^* is greater or smaller than the threshold $\rho_1 = \tilde{\rho}_1(\lambda)$ where $q_1 = 0$. ■

Bayesian Updating in the Two-Cost Case. Let us denote as $q^i(\rho, c)$ the probability with which type $i = H, L$ plays P when confronted with cost $c \in \{c_H, c_L\}$ in the W activity in period 1, and given prior beliefs $\rho_1 = \rho$. Following a recall of the first-period cost $\hat{c} = c_H$, Bayes' rule implies:

$$\frac{\hat{\rho}_2^+}{1-\hat{\rho}_2^+} = \left(\frac{\rho}{1-\rho}\right)\left(\frac{(1-\pi)q^H(\rho, c_H) + \pi(1-\nu)q^H(\rho, c_L)}{(1-\pi)q^L(\rho, c_H) + \pi(1-\nu)q^L(\rho, c_L)}\right), \tag{A.4}$$

$$\frac{\hat{\rho}_2^-}{1-\hat{\rho}_2^-} = \left(\frac{\rho}{1-\rho}\right)\left(\frac{(1-\pi)\left(1-q^H(\rho, c_H)\right) + \pi(1-\nu)\left(1-q^H(\rho, c_L)\right)}{(1-\pi)\left(1-q^L(\rho, c_H)\right) + \pi(1-\nu)\left(1-q^L(\rho, c_L)\right)}\right), \tag{A.5}$$

where $\hat{\rho}_2^+$ and $\hat{\rho}_2^-$ denote posterior after the events P and G respectively. Similarly, following a recalled cost $\hat{c} = c_L$:

$$\frac{\rho_2^+}{1-\rho_2^+} = \left(\frac{\rho}{1-\rho}\right)\left(\frac{q^H(\rho, c_L)}{q^L(\rho, c_L)}\right), \tag{A.6}$$

$$\frac{\rho_2^-}{1-\rho_2^-} = \left(\frac{\rho}{1-\rho}\right)\left(\frac{1-q^H(\rho, c_L)}{1-q^L(\rho, c_L)}\right). \tag{A.7}$$

These expressions can be simplified once it has been shown that $q^H(\rho, c_L) = 1$ and $q^L(\rho, c_H) = 0$ are dominant strategies, yielding the expressions in footnotes 32 and 33; in particular, $\rho_2^- = 0$. Note that the only case in which a posterior is undefined is that of $\hat{\rho}_2^+$ when $\nu = 1$ and the equilibrium calls for both types to play G when $c_1 = c_H$ (rules R_0, R_2 and R_{02}). Beliefs following the zero-probability event $(\hat{\sigma} = P,\ \hat{c} = c_H)$ then have to be considered, as well as refinements thereof. ■

Proof of Propositions 2 and 3. We derive here the necessary and sufficient conditions under which each rule can be sustained in equilibrium, for the general case $\nu \in (0, 1)$. We then obtain the results stated in the text by: a) letting ν tend 0 and to 1 in the formulas; b) additionally, examining the existence (and robustness to the Cho and Kreps (1987) criterion) of other equilibria (R_0, R_2, or R_{02}) that may be sustained through off-the-equilibrium-path beliefs when $\nu = 1$. (Recall that there are no unexpected events for any $\nu < 1$).

1) When is R_0 (that is, $q^H = q^L = 0$) an equilibrium in period 1?

Under R_0 the updating rules imply $\rho_2^+ = \hat{\rho}_2^+ = 1$, $\rho_2^- = 0$ and

$$\frac{\hat{\rho}_2^-}{1-\hat{\rho}_2^-} = \left(\frac{\rho_1}{1-\rho_1}\right)\chi, \tag{A.8}$$

where

$$\chi \equiv \frac{1-\pi}{1-\pi\nu} = \Pr\left[c = c_H \mid \hat{c} = c_H\right] \tag{A.9}$$

represents the "reliability" or "credibility" of ex-post excuses. The optimality conditions (9)-(10), together with the previously computed values of V_2^i, now require that:

$$\frac{c_H}{\beta_H} \geq B - b + \delta\left(\phi - V_2^H(\hat{\rho}_2^-)\right), \tag{A.10}$$

$$\frac{c_L}{\beta_L} \geq B - b + \delta\nu\,(b-a) + \delta(1-\nu)\left(b - V_2^L(\hat{\rho}_2^-)\right). \tag{A.11}$$

Let us therefore define $\bar{\rho}_1$ as the value of ρ_1 which leads to the posterior $\hat{\rho}_2^- = \rho_2^*$ in (A.8):

$$\bar{\rho}_1 \equiv \frac{\rho_2^*}{\rho_2^* + (1-\rho_2^*)\chi}. \tag{A.12}$$

Note that $\bar{\rho}_1 > \rho_1^*$ and that $\bar{\rho}_1$ is decreasing in χ. The equilibrium conditions are met when either (a) or (b) below holds:

a) $\rho_1 < \bar{\rho}_1$ and

$$\frac{c_H}{\beta_H} \geq B - b + \delta\,(\phi - a) = C_H,$$

$$\frac{c_L}{\beta_L} \geq B - b + \delta\,(b - a) = C_L.$$

b) $\rho_1 > \bar{\rho}_1$ and

$$\frac{c_L}{\beta_L} \geq B - b + \delta\nu\,(b-a)$$

• For $\nu = 0$ we therefore find that R_0 is an equilibrium in all of Regions I to IV for $\rho_1 > \bar{\rho}_1$, and in Region II for every value of ρ_1. As $\nu \to 1$, note that $\chi \to 1$ and thus $\bar{\rho}_1 \to \rho_2^*$. Consequently, R_0 is a limit equilibrium only in Regions IV (for $\rho_1 > \bar{\rho}_1$) and II (for any ρ_1). When ν is exactly equal to 1, however, $\hat{\rho}_2^+$ is unconstrained except by the monotonicity requirement, $\hat{\rho}_2^+ \geq \rho_1 = \hat{\rho}_2^-$. By choosing $\hat{\rho}_2^+ = \rho_1$, or even slightly higher, one can thus always reduce the first equilibrium condition (9) to $c_H/\beta_H \geq B - b$, which holds automatically. Thus (A.10) is no longer a requirement, meaning that R_0 is now an equilibrium as long as $c_L/\beta_L \geq C_L$. For $\rho_1 < \rho_2^*$ in Region IV, however, it fails the Cho-Kreps criterion. Indeed: (i) playing P when $c = c_H$ is strictly dominated for type β_L, by Assumption 8; (ii) with $\nu = 1$ the event $(\sigma = P,\ c = c_H)$ is perfectly observable by the period-2 self; (iii) type β_H will gain if deviating to P when $c = c_H$ identifies it as the strong type, resulting in a play of W rather than NW in period 2.

2) When is R_1 (that is, $q^H = q^L = 1$) an equilibrium in period 1?

Under R_1 the updating rules imply $\rho_2^+ = \rho_1$, $\rho_2^- =$ any $\rho' \leq \rho_1$, $\hat{\rho}_2^- = 0$ and

$$\frac{\hat{\rho}_2^+}{1-\hat{\rho}_2^+} = \left(\frac{\rho_1}{1-\rho_1}\right)\left(\frac{1}{1-\chi}\right), \tag{A.13}$$

where χ was defined in (A.9). The equilibrium conditions (9)-(10) now take the form:

$$\begin{aligned}\frac{c_H}{\beta_H} &\leq B - b + \delta\left(V_2^H(\hat{\rho}_2^+) - a\right), \\ \frac{c_L}{\beta_L} &\leq B - b + \delta\nu\left(V_2^L(\rho_1) - V_2^L(\rho')\right) + \delta(1-\nu)\left(V_2^L(\hat{\rho}_2^+) - a\right).\end{aligned}$$

Given Assumption 6, the first condition requires that $c_H/\beta_H \leq B - b + \delta(\phi - a) = C_H$ and $\hat{\rho}_2^+ \geq \rho_2^*$. Define therefore $\underline{\rho}_1$ as value of ρ_1 which leads to the posterior $\hat{\rho}_2^+ = \rho_2^*$ in (A.13):

$$\underline{\rho}_1 \equiv \frac{\rho_2^*}{\rho_2^* + (1-\rho_2^*)/(1-\chi)}. \tag{A.14}$$

Note that $\underline{\rho}_1 < \rho_2^*$, and that $\underline{\rho}_1$ is decreasing in χ. We must have $\rho_1 > \underline{\rho}_1$, so the second equilibrium condition takes the form:

$$\frac{c_L}{\beta_L} \leq B - b + \delta\nu\left(V_2^L(\rho_1) - V_2^L(\rho')\right) + \delta(1-\nu)(b-a). \tag{A.15}$$

For $\rho_1 > \rho_2^*$, it can be met with $\rho' \leq \rho_1$ as long as

$$\frac{c_L}{\beta_L} \leq B - b + \delta(b-a) = C_L.$$

For $\rho_1 \in (\underline{\rho}_1, \rho_2^*)$ the second term in (A.15) is zero, so the requirement becomes:

$$\frac{c_L}{\beta_L} \leq B - b + \delta(1-\nu)(b-a).$$

To summarize, first it must be that $c_H/\beta_H \leq C_H$. Second, when $c_L/\beta_L < B - b + \delta(1-\nu)(b-a)$ this equilibrium exists for all $\rho \in (\underline{\rho}_1, 1)$; when $B - b + \delta(1-\nu)(b-a) < c_L/\beta_L < B - b + \delta(b-a)$ it exists for all $\rho \in (\rho_2^*, 1)$. In all other cases it does not exist.

• In particular, when $\nu = 0$ the equilibrium exists only in Region III, for $\rho_1 > \underline{\rho}_1$. When $\nu = 1$, implying $\underline{\rho}_1 = 0$, it exists in Region III for $\rho > \rho_2^*$.

3) When is R_2 (that is, $q^H = 0,\ q^L = 1$) an equilibrium in period 1?

Under R_2 the updating rules imply $\rho_2^+ = \rho_1$, $\rho_2^- =$ any $\rho' \leq \rho_1$ and $\hat{\rho}_2^+ = \hat{\rho}_2^- = \rho_1$. The equilibrium conditions (9)-(10) now take the form $c_H/\beta_H \geq B - b$, which always holds, and

$$\frac{c_L}{\beta_L} \leq B - b + \delta\nu\left(V_2^L(\rho_1) - V_2^L(\rho')\right).$$

This requires that $\rho_1 > \rho_2^* > \rho'$; since $\rho' \leq \rho_1$ is unconstrained, only the first of these two inequalities matters. Finally, it must be that:

$$\frac{c_L}{\beta_L} \leq B - b + \delta\nu(b-a).$$

• With $\nu = 0$, R_2 is therefore never an equilibrium. With either $\nu \to 1$ or $\nu = 1$, it is an equilibrium for $c_L/\beta_L \leq C_L$ (Regions I and III), provided that $\rho_1 > \rho_2^*$; note that in this equilibrium (9) is not binding when $\nu < 1$, and thus a fortiori not when $\nu = 1$.

4) When is R_3 (that is, $q^H = 1,\ q^L = 0$) **an equilibrium in period 1?**

Under R_3 the updating rules imply $\rho_2^+ = \hat{\rho}_2^+ = 1$, $\rho_2^- = \hat{\rho}_2^- = 0$. The equilibrium conditions (9)-(10) now take the form:

$$\begin{aligned} \frac{c_H}{\beta_H} &\leq B - b + \delta\,(\phi - a) = C_H, \\ \frac{c_L}{\beta_L} &\geq B - b + \delta\,(b - a) = C_L. \end{aligned}$$

• Thus, whether for $\nu = 0$ or $\nu = 1$, R_3 is an equilibrium in Region IV, for all values of ρ_1.

5) When is R_{02} (that is, $q^H = 0,\ q^L \in (0,1)$) **an equilibrium in period 1?**

Under R_{02} the updating rules imply $\rho_2^- = 0$ and

$$\frac{\rho_2^+}{1-\rho_2^+} = \frac{\hat{\rho}_2^+}{1-\hat{\rho}_2^+} = \left(\frac{\rho_1}{1-\rho_1}\right)\left(\frac{1}{q^L}\right),$$

$$\frac{\hat{\rho}_2^-}{1-\hat{\rho}_2^-} = \left(\frac{\rho_1}{1-\rho_1}\right)\left(\frac{1-\pi}{1-\pi+\pi(1-\nu)\,(1-q^L)}\right).$$

Conditions (9)-(10) now take the form:

$$\begin{aligned} \frac{c_H}{\beta_H} &\geq B - b + \delta\left(V_2^H(\hat{\rho}_2^+) - V_2^H(\hat{\rho}_2^-)\right), && \text{(A.16)} \\ \frac{c_L}{\beta_L} &= B - b + \delta\nu\left(V_2^L(\rho_2^+) - a\right) + \delta(1-\nu)\left(V_2^L(\rho_2^+) - V_2^L(\hat{\rho}_2^-)\right). && \text{(A.17)} \end{aligned}$$

The second one cannot hold (except with measure zero) unless either ρ_2^+ or $\hat{\rho}_2^-$ equals ρ_2^* .

Case 1: $\rho_2^+ = \rho_2^*$, which uniquely defines q^L as long as $\rho_1 < \rho_2^*$. Conditions (9)-(10) become:

$$\begin{aligned} \frac{c_H}{\beta_H} &\geq B - b + \delta p_2(\rho_2^*)\,(\phi - a)\,, \\ \frac{c_L}{\beta_L} &= B - b + \delta p_2(\rho_2^*)\,(b - a)\,. \end{aligned}$$

Abbreviating $p_2(\rho_2^*)$ as p_2^*, the second condition yields $p_2^* = (c_L/\beta_L - B + b)\,/\,(\delta(b-a))$, so the equilibrium requirements finally become:

$$\begin{aligned} \frac{c_L}{\beta_L} &\leq B - b + \delta\,(b - a) = C_L, && \text{(A.18)} \\ \frac{c_H}{\beta_H} &\geq B - b + \left(\frac{c_L}{\beta_L} - B + b\right)\left(\frac{\phi - a}{b - a}\right). && \text{(A.19)} \end{aligned}$$

In the $(c_L/\beta_L,\, c_H/\beta_H)$ plane, the boundary for the latter inequality is the line $\mathfrak{L}_1$, with slope $(\phi - a)/(b - a)$, that goes from the point $(B - b, B - b)$ to the point $(B - b + \delta\,(b - a)\,, B - b + \delta\,(\phi - a)) = (C_L, C_H)$, thus separating regions III$^-$ and III$^+$ as indicated on Figure 4.

Case 2: $\hat{\rho}_2^- = \rho_2^*$, which by the updating rules uniquely defines q^L as long as

$$\rho_2^* < \rho_1 < \frac{\rho_2^*}{\rho_2^* + (1 - \rho_2^*)\chi} = \bar{\rho}_1. \tag{A.20}$$

The equilibrium conditions then become:

$$\begin{aligned} \frac{c_H}{\beta_H} &\geq B - b + \delta\,(1 - p_2^*)\,(\phi - a)\,, \\ \frac{c_L}{\beta_L} &= B - b + \delta\,[\nu + (1 - \nu)\,(1 - p_2^*)]\,(b - a)\,. \end{aligned}$$

The latter yields: $1 - p_2^* = [(c_L/\beta_L - B + b)\,/\,(\delta\,(b - a)) - \nu]\,/\,(1 - \nu)$ as long as

$$B - b + \nu\delta(b - a) < c_L/\beta_L < B - b + \delta\,(b - a) = C_L.$$

The first condition then requires:

$$\frac{c_H}{\beta_H} \geq B - b + \left(\frac{c_L/\beta_L - B + b - \nu\delta\,(b - a)}{1 - \nu}\right)\left(\frac{\phi - a}{b - a}\right). \tag{A.21}$$

In the $(c_L/\beta_L,\, c_H/\beta_H)$ plane, the boundary for the second one is the line $\mathfrak{L}_2$, with slope $(\phi - a)/\left[(1 - \nu)(b - a)\right]$, that goes from the point $(B - b + \delta\nu(b - a), B - b)$ to the point $(B - b + \delta\,(b - a)\,, B - b + \delta\,(\phi - a))$.

• For $\nu = 0$, R_{02} therefore exists in Regions I and III$^+$ for $\rho_1 < \rho_2^*$ (Case 1) as well as for $\rho_2^* < \rho_1 < \bar{\rho}_1$ (Case 2), and thus for all $\rho_1 < \bar{\rho}_1$. As $\nu \to 1$ we have $\bar{\rho}_1 \to \rho_2^*$, so it exists in Regions I and III$^+$ for $\rho_1 < \rho_2^*$ (Case 1). When ν is exactly equal to 1, $\hat{\rho}_2^+$ is again unconstrained except by the monotonicity requirement, $\hat{\rho}_2^+ \geq \rho_1$. Case 2 is still inapplicable since $\hat{\rho}_2^- = \rho_1$, while in Case 1 one can again choose $\hat{\rho}_2^+$ so as to reduce (9) to $c_H/\beta_H \geq B - b$, which always holds. The only binding equilibrium condition is then (A.18), together with $\rho_1 < \rho_2^*$ which is required for $\rho_2^+ = \rho_2^*$ to have a solution in q^L. Thus R_{02} exists in all of Regions I and III when $\rho_1 < \rho_2^*$. In the latter, however, it fails the Cho-Kreps criterion; the proof is identical to that given earlier to eliminate R_2 from Region IV when $\rho_1 < \rho_2^*$.

6) When is R_{03} (that is, $q^H \in (0,1),\ q^L = 0$) an equilibrium in period 1?

Under R_{03} the updating rules imply $\rho_2^+ = 1$, $\rho_2^- = 0$, $\hat{\rho}_2^+ = 1$ and

$$\frac{\hat{\rho}_2^-}{1 - \hat{\rho}_2^-} = \left(\frac{\rho_1}{1 - \rho_1}\right)\left(\frac{(1 - \pi)\left(1 - q^H\right)}{1 - \pi + \pi(1 - \nu)}\right) = \chi\left(1 - q^H\right)\left(\frac{\rho_1}{1 - \rho_1}\right).$$

The equilibrium conditions (9)-(10) now take the form:

$$\begin{aligned} \frac{c_H}{\beta_H} &= B - b + \delta\left(\phi - V_2^H(\hat{\rho}_2^-)\right), \\ \frac{c_L}{\beta_L} &\geq B - b + \delta\nu\,(b-a) + \delta(1-\nu)\left(b - V_2^L(\hat{\rho}_2^-)\right). \end{aligned}$$

The first condition requires that $\hat{\rho}_2^- = \rho_2^*$, which uniquely determines q^H as long as $\rho_1 > \bar{\rho}_1$ defined earlier in (A.12). Then, $\phi - V_2^H(\hat{\rho}_2^-) = (1-p_2^*)(\phi - a)$, or $1 - p_2^* = (c_H/\beta_H - B + b)\,/\,(\delta(\phi - a))$, requiring that:

$$c_H/\beta_H < B - b + \delta\,(\phi - a) = C_H.$$

The second equilibrium condition then becomes:

$$\frac{c_L}{\beta_L} \geq B - b + \delta\nu\,(b-a) + (1-\nu)\,(c_H/\beta_H - B + b)\left(\frac{b-a}{\phi - a}\right).$$

In the $(c_L/\beta_L,\, c_H/\beta_H)$ plane, the boundary for this inequality is again the line $\mathfrak{L}_2$, with slope $(\phi - a)/[(1-\nu)(b-a)]$, that goes from the point $(B - b + \delta\nu(b-a), B-b)$ to the point $(B - b + \delta\,(b-a)\,, B - b + \delta\,(\phi - a))$.

- For $\nu = 0$, R_{03} therefore exists in Regions III$^-$ and IV for $\rho_1 > \bar{\rho}_1$. For $\nu = 1$, in which case $\bar{\rho}_1 = \rho_2^*$, it exists in Region IV only, for $\rho_1 > \rho_2^*$.

7) When is R_{12} (that is, $q^H = 1,\ q^L \in (0,1)$) an equilibrium in period 1?

Under R_{12} the updating rules imply $\rho_2^+ = \rho_1$, $\rho_2^- =$ any $\rho' \leq \rho_1$, and

$$\begin{aligned} \frac{\hat{\rho}_2^-}{1 - \hat{\rho}_2^-} &= \left(\frac{\rho_1}{1-\rho}\right)(1 - q^H), \\ \frac{\hat{\rho}_2^+}{1 - \hat{\rho}_2^+} &= \left(\frac{\rho_1}{1-\rho_1}\right)\left(\frac{(1-\pi)q^H + \pi(1-\nu)}{\pi(1-\nu)}\right). \end{aligned}$$

Conditions (9)-(10) now take the form:

$$\begin{aligned} \frac{c_H}{\beta_H} &= B - b + \delta\left(V_2^H(\hat{\rho}_2^+) - V_2^H(\hat{\rho}_2^-)\right), \\ \frac{c_L}{\beta_L} &\leq B - b + \delta\nu\left(V_2^L(\rho_1) - V_2^L(\rho')\right) + \delta(1-\nu)\left(V_2^L(\hat{\rho}_2^+) - V_2^L(\hat{\rho}_2^-)\right). \end{aligned}$$

The first one requires either Case 1 or Case 2 below.

Case 1: $\hat{\rho}_2^- = \rho_2^*$, which uniquely defines q^H as long as $\rho_1 > \rho_2^*$. Then $1 - p_2^* = (c_H/\beta_H - B + b)\,/\,(\delta(\phi - a))$, requiring

$$c_H/\beta_H < B - b + \delta\,(\phi - a) = C_H.$$

The second equilibrium condition then becomes:

$$\frac{c_L}{\beta_L} \leq B - b + \delta\nu\left(b - V_2^L(\rho')\right) + (1-\nu)\left(c_H/\beta_H - B + b\right)\left(\frac{b-a}{\phi-a}\right).$$

This can be satisfied with $\rho' \leq \rho_1$ as long as

$$\frac{c_L}{\beta_L} \leq B - b + \delta\nu\left(b - a\right) + (1-\nu)\left(c_H/\beta_H - B + b\right)\left(\frac{b-a}{\phi-a}\right).$$

In the $(c_L/\beta_L,\, c_H/\beta_H)$ plane, the boundary for the latter inequality is again the line $\mathfrak{L}_2$, with slope $(\phi-a)/[(1-\nu)(b-a)]$, that goes from the point $(B-b+\delta\nu\,(b-a)\,, B-b)$ to the point $(B-b+\delta\,(b-a)\,, B-b+\delta\,(\phi-a))$.

Case 2: $\hat{\rho}_2^+ = \rho_2^*$, which then uniquely defines q^H as long as $\underline{\rho}_1 < \rho_1 < \rho_2^*$. Then,

$$\begin{aligned}\frac{c_H}{\beta_H} &= B - b + \delta p_2^*\left(\phi - a\right),\\ \frac{c_L}{\beta_L} &\leq B - b + \delta\nu\times 0 + \delta(1-\nu)p_2^*\left(b-a\right).\end{aligned}$$

which uniquely determines p_2^* as long as

$$\begin{aligned}\frac{c_H}{\beta_H} &\leq B - b + \delta\left(\phi - a\right) = C_H,\\ \frac{c_L}{\beta_L} &\leq B - b + (1-\nu)\left(c_H/\beta_H - B + b\right)\left(\frac{b-a}{\phi-a}\right).\end{aligned}$$

In the $(c_L/\beta_L,\, c_H/\beta_H)$ plane the boundary for the latter inequality is the line $\mathfrak{L}_3$, with slope $(\phi-a)/[(1-\nu)\delta(b-a)]$ (same as for $\mathfrak{L}_2$) that goes from the point $(B-b, B-b)$ to the point $(B-b+\delta(1-\nu)\,(b-a)\,, B-b+\delta\,(\phi-a))$.

- Putting together Cases 1 and 2, we see that when $\nu = 0$ R_{12} exists only in Region III$^+$ for $\rho > \rho_2^*$ (Case 1) as well as for $\underline{\rho}_1 < \rho_1 < \rho_2^*$ (Case 2); hence, for all $\rho_1 > \underline{\rho}_1$. When $\nu = 1$ it exists in all of Region III for $\rho > \rho_2^*$ (Case 1).

8) When is R_{13} (that is, $q^H = 1,\ q^L \in (0,1)$) an equilibrium in period 1?

Under R_{13} the updating rules imply $\rho_2^- = \hat{\rho}_2^- = 0$,

$$\begin{aligned}\frac{\rho_2^+}{1-\rho_2^+} &= \left(\frac{\rho_1}{1-\rho_1}\right)\left(\frac{1}{q^L}\right),\\ \frac{\hat{\rho}_2^+}{1-\hat{\rho}_2^+} &= \left(\frac{\rho_1}{1-\rho_1}\right)\left(\frac{1-\pi+\pi(1-\nu)}{\pi(1-\nu)q^L}\right) = \left(\frac{\rho_1}{1-\rho_1}\right)\left(\frac{1}{q^L}\right)\left(\frac{1}{1-\chi}\right).\end{aligned}$$

The equilibrium conditions (9)-(10) now take the form:

$$\begin{aligned}\frac{c_H}{\beta_H} &\leq B - b + \delta\left(V_2^H(\hat{\rho}_2^+) - a\right),\\ \frac{c_L}{\beta_L} &= B - b + \delta\nu\left(V_2^L(\rho_2^+) - a\right) + \delta(1-\nu)\left(V_2^L(\hat{\rho}_2^+) - a\right).\end{aligned}$$

The second one cannot hold (except with measure zero) unless either ρ_2^+ or $\hat{\rho}_2^+$ equals ρ_2^* .

Case 1: $\rho_2^+ = \rho_2^*$, which then uniquely defines q^L as long as $\rho_1 < \rho_2^*$. Since $\hat{\rho}_2^+ > \rho_2^+$ always, the equilibrium conditions then become

$$\begin{aligned}\frac{c_H}{\beta_H} &\leq B - b + \delta(\phi - a) = C_H,\\ \frac{c_L}{\beta_L} &= B - b + \delta\left[\nu p_2^* + 1 - \nu\right](b - a).\end{aligned}$$

Hence $p_2^* = \left[c_L/\beta_L - B + b - \delta(1-\nu)(b-a)\right] / \left[\delta\nu(b-a)\right]$, requiring:

$$B - b + \delta(1-\nu)(b-a) < c_L/\beta_L < B - b + \delta(b-a) = C_L.$$

Case 2: $\hat{\rho}_2^+ = \rho_2^*$, which then uniquely defines q^L, as long as $\rho_1 < \underline{\rho}_1$ defined in (A.14). Since $\hat{\rho}_2^+ > \rho_2^+$ always, the two conditions then become:

$$\begin{aligned}\frac{c_H}{\beta_H} &\leq B - b + \delta p_2^*(\phi - a),\\ \frac{c_L}{\beta_L} &= B - b + \delta(1-\nu)p_2^*(b-a).\end{aligned}$$

The latter condition determines p_2^* uniquely, as long as

$$\frac{c_L}{\beta_L} \leq B - b + \delta(1-\nu)(b-a).$$

Finally, the first condition requires

$$\frac{c_H}{\beta_H} \leq B - b + \left(\frac{c_L/\beta_L - B + b}{1-\nu}\right)\left(\frac{\phi - a}{b - a}\right),$$

In the $(c_L/\beta_L,\, c_H/\beta_H)$ plane, the boundary is again the line $\mathfrak{L}_3$, with slope $(\phi-a)/[(1-\nu)(b-a)]$, that goes from the point $(B-b, B-b)$ to the point $(B-b+\delta(1-\nu)(b-a), B-b+\delta(\phi-a))$.

• Therefore, when $\nu = 0$, R_{13} exists only in Region III$^-$ only for $\rho < \underline{\rho}_1$(Case 2). When $\nu = 1$ it exists in all of Region III for $\rho_1 < \rho_2^*$ (Case 1).

9) When is R_{01} (that is, $q^H \in (0,1),\ q^L \in (0,1)$) an equilibrium in period 1?

Under R_{01} the updating rules imply $\rho_2^- = 0$ and

$$\begin{aligned}
\frac{\rho_2^+}{1-\rho_2^+} &= \left(\frac{\rho_1}{1-\rho_1}\right)\left(\frac{1}{q^L}\right) \\
\frac{\hat{\rho}_2^+}{1-\hat{\rho}_2^+} &= \left(\frac{\rho_1}{1-\rho_1}\right)\left(\frac{(1-\pi)q^H + \pi(1-\nu)}{\pi(1-\nu)q^L}\right) \\
\frac{\hat{\rho}_2^-}{1-\hat{\rho}_2^-} &= \left(\frac{\rho_1}{1-\rho_1}\right)\left(\frac{(1-\pi)\left(1-q^H\right)}{1-\pi+\pi(1-\nu)\left(1-q^L\right)}\right)
\end{aligned}$$

Note that $\hat{\rho}_2^+ > \rho_2^+ > \rho_1 > \hat{\rho}_2^-$. The equilibrium conditions are then:

$$\begin{aligned}
\frac{c_H}{\beta_H} &= B - b + \delta\left[V_2^H(\hat{\rho}_2^+) - V_2^H(\hat{\rho}_2^-)\right], \\
\frac{c_L}{\beta_L} &= B - b + \delta\nu\left[V_2^L(\rho_2^+) - a\right] + \delta(1-\nu)\left[V_2^L(\hat{\rho}_2^+) - V_2^L(\hat{\rho}_2^-)\right].
\end{aligned}$$

The first condition cannot be an equality (except with measure zero in the parameter space) unless either $\hat{\rho}_2^+$ or $\hat{\rho}_2^-$ is equal to ρ_2^* ; in that case, the equality determines at most one suitable p_2^*. The second condition cannot be an equality unless either ρ_2^+ or $\hat{\rho}_2^+$ or $\hat{\rho}_2^-$ is equal to ρ_2^*; in either case, the equality again determines at most one suitable p_2^*. These two values of p_2^* do not coincide, except with measure zero in the $(c_L/\beta_L,\ c_H/\beta_H)$ space. Thus an equilibrium of this type cannot exist, as no single mixing strategy can make both types indifferent.

To conclude the proofs of Propositions 2 and 3, it just remains to check that the equilibrium indicated in bold in each of the areas of Figures 3 and 4 where multiplicity occurs is the one preferred by the β_H type, and when $b \geq a$ by the β_L type as well. This is straightforward. ■